Ising spin glass models versus Ising models: an effective mapping at high temperature I. General result

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Abstract. We show that, above the critical temperature, if the dimension D of a given Ising spin glass model is sufficiently high, its free energy can be effectively expressed through the free energy of a related Ising model. When, in a large sense, $D=\infty$, in the paramagnetic phase and on its boundary the mapping is exact. In this limit the method provides a general and simple rule to obtain exactly the upper phase boundaries. We provide even simple effective rules to find crossover surfaces and correlation functions. We apply the mapping to several spin glass models.

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1. Introduction

The spin glass model represents one of the most difficult challenges in physics (see Ref. [1] and referred articles). When compared to a non random model, the spin glass model turns out to be hugely more difficult to solve. In fact, to solve a spin glass model, one should be able to study, at least in principle, either a non random model, but with non uniform frozen couplings, or, as appears evident in the framework of the replica approach, a system of n coupled uniform non random models, extrapolating some suitable derivative in the limit $n \to 0$. From this point of view it seems fairly hard that a spin glass might be solved simply in terms of a mapping with a non random and uniform model. Nevertheless, in this paper we show that, whereas in the most rich and complex low temperature phases such a mapping is impossible, when the dimensionality D of the model tends to infinite, in the paramagnetic phase and on its boundary, the given spin glass model can be easily solved by simply considering a suitable corresponding non random Ising model: "the related Ising model". More precisely, we shall show that, when in a large sense the dimensionality D is infinite, as happens, *e.g.*, but not only, in mean field models and in generalized tree-like structures (allowing also for the presence of loops), once the phase boundary of the related Ising model is known, it can be immediately used in the spin glass model to find exactly the region of the paramagnetic phase (P) and its boundaries with the other phases: ferromagnetic disordered (F), spin glass (SG), or antiferromagnetic disordered (AF).

The derivation of this mapping, "spin glass" $\to$ "related Ising model", will be obtained in the framework of a replica approach, which however differs from the standard one [1]. As we shall show, the use of replicas in the high temperature expansion of the free energy leads to a different procedure in which there is no functional to be extremized and, furthermore, when D is large, at high temperature a simple combinatorial approximation applies which, in the limit $D \to \infty$, becomes exact. We point out that, unlike the standard replica approach, in the proof for this mapping we do not need to rely on any ansatz concerning the choice of finding stationary points. Therefore our proof, up to question of the analytic continuation n integer $\to n$ real, is exact [2]-[6].

Previous uses of the high temperature expansion to study spin glass models go back to [7] for the Sherrignton-Kirkpatrick model (SK) [8], and to [9] and [10] for the Ising spin glass models on hypercubic lattices with simmetric disorder, where very accurate results were found for small and high dimensions in [9] and [10], respectively (we cite also the Ref. [11] and the Refs. [12, 13] for the Potts glass model). However, as will be shown, our use of the high temperature expansion of the partition function is quite different. We do not use this expansion to directly calculate the averages over the disorder of a specific model, we use instead the expansion to find a general link between the spin glass and a suitable related Ising model. Once this link has been established, the singularities of the related Ising model will provide the singularities of the spin glass, signaling the phase transitions in the sense of Lee and Yang [14].

The practical potential applications of this mapping are remarkable. In fact, though the mapping may be used to analyze only the regions at high temperature, including the upper phase boundary and, within a certain approximation, the crossover surfaces and the correlation functions, as will be clear, its generality and simplicity make the approach particularly suitable to face those infinite dimensional spin glass models whose great complexity, *e.g.* due to the presence of too many parameters, can make difficult, or avoids at all, the application of the standard methods for spin

glasses even in the easier paramagnetic phase. In our approach instead, the model to be solved is the related Ising model which, as a non random model, turns out to be hugely easier to solve, analytically and/or numerically. In this work (part I) we will show this in several non trivial known examples, whereas in a forthcoming paper (part II) the mapping will be applied for studing the Ising spin glass model on general graphs and networks.

The paper is organized as follows. In Secs. 2 and 3 we introduce the models and the condition under which the mapping holds. In Sec. 4, after introducing the definition of the related Ising model, we present formally the mapping, subdiving the results in the subsections 4.1 (same disorder for any bond), 4.2 (generalization), 4.3 (upper line of the phase diagram). As a by-product, in the further subsection 4.4, via analytic continuation, we, improperly, force the mapping to find crossover surfaces and correlation functions. We stress that this extension of the mapping is not exact, but provide however a first effective insight about the physics of the model.

The rest of the article until Sec. 11 is devoted to the proof of the mapping. In Sec. 5 we give some preliminary ideas, whereas in Sec. 6 we recall the high temperature expansion for a general Ising (even non uniform) model. In Sec. 7, starting form the high temperature expansion, we carry out formally the average over the disorder to be used in Sec. 8 for calculating the free energy. In Sec. 9 we specialize the expansion for a centered measure, whereas in the following subsection 9.1 the basic approximation in high dimension will be introduced and used to find the mapping for a hypercube lattice. In Sec. 10 and in the following subsection we generalize the mapping to any measure; finally, in Sec. 11 we extend the mapping to any model whose dimension, in a large sense, turns out to be infinite, as happens, in particular, but not only, in tree-like structures and generalized tree-like structures.

The sections 12 and 13 are devoted to some applications in the case of D finite and infinite, respectively. We anticipate that our approach takes into account only the leading term of a $1/D$ expansion, so that the applications in the finite dimensional case are basically meant to show how the method works.

Finally, conclusions and some outlooks are reported in Sec. 14. The paper is equipped with three appendix.

2. Models

Let us consider a D dimensional hypercube of side L, $\Lambda = \{1, \ldots, L\}^D$, and the set of the bonds b connecting two first neighbors sites

$$\Gamma \equiv \{b = (i_b, i_b) : i_b, j_b \in \Lambda,\ i_b \text{ and } j_b \text{ first neighbors},\ i_b < j_b\}. \tag{1}$$

We will indicate with N the total number of points, $N = L^D$. More in general, we shall consider also systems over a graph. Given a graph g of N vertices, the set of links will be defined through the adjacency matrix of the graph, $g_{i,j} = 0, 1$:

$$\Gamma \equiv \{b = (i_b, i_b) : i_b, j_b \in g,\ g_{i_b, j_b} = 1,\ i_b < j_b\}. \tag{2}$$

The set of links of the fully connected graph will be indicated with Γ_f:

$$\Gamma_f \equiv \{b = (i_b, i_b) : i_b, j_b = 1, \ldots, N,\ i_b < j_b\}. \tag{3}$$

The Hamiltonian of the spin glass with two-body interactions can be written as

$$H\left(\{\sigma_i\}; \{J_b\}; \{h_i\}\right) \equiv -\sum_{b\in\Gamma} J_b \tilde{\sigma}_b + \sum_{i=1}^{N} h_i \sigma_i, \tag{4}$$

where the h_i's are arbitrary external fields, the J_b's are quenched couplings, σ_i is an Ising variable at the site i, and $\tilde{\sigma}_b$ stays for the product of two Ising variables, $\tilde{\sigma}_b = \sigma_{i_b}\sigma_{j_b}$, with i_b and j_b such that $b = (i_b, j_b)$.

The free energy F is defined by

$$-\beta F \equiv \int d\mathcal{P}\left(\{J_b\}\right) \log\left(Z\left(\{J_b\};\{h_i\}\right)\right), \tag{5}$$

where $Z\left(\{J_b\};\{h_i\}\right)$ is the partition function of the quenched system

$$Z\left(\{J_b\};\{h_i\}\right) = \sum_{\{\sigma_b\}} e^{-\beta H(\{\sigma_i\};\{J_b\};\{h_i\})}, \tag{6}$$

and $d\mathcal{P}\left(\{J_b\}\right)$ is a product measure over all the possible bonds b given in terms of normalized measures $d\mu_b \geq 0$ (we are considering a general measure $d\mu_b$ allowing also for a possible dependence on the bonds)

$$d\mathcal{P}\left(\{J_b\}\right) \equiv \prod_{b\in\Gamma_f} d\mu_b\left(J_b\right), \quad \int d\mu_b\left(J_b\right) = 1. \tag{7}$$

Among the measures most considered in literature, we cite in particular the Gaussian measure

$$\frac{d\mu\left(J_b\right)}{dJ_b} = \frac{1}{\sqrt{2\pi\tilde{J}^2}} \exp\left(-J_b^2/\tilde{J}^2\right), \tag{8}$$

and the "plus-minus" measure

$$\frac{d\mu(J_b)}{dJ_b} = \frac{1}{2}\delta(J - J_b) + \frac{1}{2}\delta(J + J_b), \tag{9}$$

where $\tilde{J}^2$ and J represent disorder parameters. We will take the Boltzmann constant $K_B = 1$. A generic inverse critical temperature of the spin glass model, if any, will be indicated with β_c; sometimes we will use the symbol $< A >$ for the quenched thermal average of the quantity A (fixed values of the couplings $\{J_b\}$); finally the density free energy in the thermodynamic limit will be indicated with $f = f(\beta)$

$$f(\beta) \equiv \lim_{N\to\infty} F(\beta)/N. \tag{10}$$

3. Dimensionality - Condition for the mapping

Our mapping will be first derived in a D dimensional hypercubic lattice by applying an approximation becoming exact in the limit $D \to \infty$. Successively, we will consider more general structures, whose dimension, in a large sense, goes to infinite as well. For a hypercube lattice the dimension D is related to the number of first neighbors of a vertex, $2D$, so that, in this case, and in others in which the numbers of first neighbors is proportional to D, $D \to \infty$ if the number of first neighbors goes to infinite For these cases, we shall say that the set of links Γ is *infinite dimensional in a strict sense*. For other structures, a usual definition of dimensionality is

$$D \equiv \lim_{l\to\infty} \frac{\log\left(N(l)\right)}{\log\left(l\right)}, \tag{11}$$

where $N(l) = 1 + m_1 + \ldots + m_l$ represents the total number of vertices within l steps of an arbitrary fixed root vertex 0, m_k being the number of vertices at distance k from the root. We will not make use of this definition for the dimension, but here we

just observe what follows. When applied to a hypercube lattice, Eq. (11) returns the true dimension of the lattice D, whereas in the case of a tree with an average degree greater than 1, Eq. (11) gives $D = \infty$. As an heuristic argument we can say that, given a structure Γ, the hypercube lattice most similar to Γ must have a dimension D given by Eq. (11); therefore, if, for Γ, D of Eq. (11) goes to infinite, the mapping must hold. However, we will not need to consider this heuristic argument. In Sec. 11, we will work out directly the derivation of the mapping for tree-like structures, little differences being involved with respect to the hypercubic lattice. Furthermore, we shall now introduce the most general condition under which the mapping is exact, stressing that the tree-like structure is not necessary.

Let us define a path as a succession of first neighbor vertices connected by different bonds of Γ. A bond has length 1. The paths can be open or closed (loops). As will be clear later, depending on Γ and near the limit $\beta \to \beta_c^-$, the paths giving a finite contribution to the partition function Z are infinitely long paths which can be: closed, as for hypercubic lattices; open, as for tree-like structures; and both closed and open, as for generalized tree-like structures. We say that Γ has a generalized tree-like structure if for any vertex there is at most a finite number of loops.

What will emerge in subsection 9.1 and Sec. 11, is that

when, in the thermodynamic limit, for the given set Γ, the total number of infinitely long paths per vertex goes to infinite and, choosing randomly two of them, the probability that they overlap each other for an infinite number of bonds goes to zero, in the limit $\beta \to \beta_c^-$, the mapping becomes exact.

We shall say that such a Γ is *infinite dimensional in the large sense*. It is immediate to verify that in the case of tree-like structures the above condition is trivially satisfied and similarly for generalized tree-like structures. In general, given Γ, the absence of loops, or the presence of a finite number of loops per vertex, turns out to be a sufficient condition for satisfying the above requirement. Note however that this is not a necessary condition. Loops can also be massively present; what is necessary is only that loops overlap, either each other or with other paths, only partially (see Fig. 1). Of course such a behavior does not happen in a hypercube lattice of finite dimension D. It is in fact easy to see that, in this case, the total number of paths of length l per vertex grows as $c(l) \sim \exp(\mu_D l)$, with μ_D a suitable constant [21], whereas the maximum number of paths which after l steps do not share any other bond grows only as $c_i(l) \sim l^D$. Therefore, if D is finite, in the limit $l \to \infty$, we see that the condition cannot be satisfied.

4. The mapping

Given a spin glass model trough Eqs. (2-7), we define, on the same set of links Γ, its *related Ising model* trough the following Ising Hamiltonian

$$H_I\left(\{\sigma_i\};\{J_b\};\{h_i\}\right) \equiv -\sum_{b\in\Gamma} J_b^{(I)}\tilde{\sigma}_b + \sum_{i=1}^{N} h_i\sigma_i \tag{12}$$

where the Ising couplings $J_b^{(I)}$'s have non random values such that $\forall\ b, b' \in \Gamma$

$$J_{b'}^{(I)} = J_b^{(I)} \quad \text{if} \quad d\mu_{b'} \equiv d\mu_b, \tag{13}$$

$$J_b^{(I)} \neq 0 \quad \text{if} \int d\mu_b(J_b) J_b \neq 0 \quad \text{or} \quad \int d\mu_b(J_b) J_b^2 > 0. \tag{14}$$

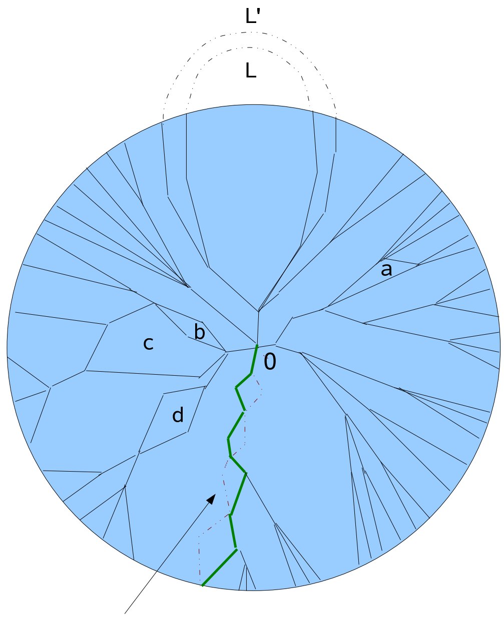

Figure 1. Example to clarify the condition given in Sec. 3. From the root vertex 0 depart several infinitely long paths. For simplicity, in the figure it is understood that, except for the chain indicated by an arrow, besides the loops a, b, c, d, L and L', there are no other loops connected to the root 0. It is also understood that some branching of the paths keeps on also out of the circle. From each of the loops a, b, c, and d, and from the other branches without loops, pass an infinite number of infinitely long paths, but the probability that two of them, randomly chosen, overlap for an infinite number of bonds, goes to 0. Through the chain of loops (infinite chain) pass also an infinite number of infinitely long paths, but in this case, the probability that, two of them randomly chosen, overlap for an infinite number of bonds is not 0. A way to generate, from the above graph, a graph infinite dimensional in the large sense, consists in deleting, *e.g.*, all the infinite bonds of the chain indicated with dashed segments. The loops L and L' are two example of infinite closed paths. Note that, since they share only a finite number of bonds each other, their presence do not alter the condition of Sec. 3.

In the following a suffix I over quantities such as H_I, F_I, f_I, etc..., or $J_b^{(I)}$, $\beta_c^{(I)}$, etc..., will be referred to the related Ising system with Hamiltonian (12). We can always split the free energy of the quenched model as follows

$$-\beta F = \sum_{b\in\Gamma} \int d\mu_b \log\left(2\cosh(\beta J_b)\right) + \sum_{i=1}^{N} \log\left(2\cosh(\beta h_i)\right) + \phi, \quad (15)$$

ϕ being the high temperature part of the free energy, and similarly for the related Ising model. Let φ be the density of ϕ in the thermodynamic limit

$$\varphi \equiv \lim_{N\to\infty} \phi/N, \quad (16)$$

and similarly for φ_I, the high temperature part of the free energy density of the related Ising model defined through Eqs. (12-14). As is known, the high temperature part φ (φ_I) can be expressed in terms of the quantities $z_b = \tanh(\beta J_b)$ ($z_b^{(I)} = \tanh(\beta J_b^{(I)})$) and $z_i = \tanh(\beta h_i)$, *i.e.*, the adimensional parameters of the Ising high temperature

expansion:

$$\varphi \;\; = \varphi\left(\{\tanh(\beta J_b)\};\{\tanh(\beta h_i)\}\right), \qquad (17)$$

$$\varphi_I = \varphi_I\left(\{\tanh(\beta J_b^{(I)})\};\{\tanh(\beta h_i)\}\right), \qquad (18)$$

Hereafter, if not explicitly said, we set $h_i \equiv 0$. By varying in the Ising Hamiltonian (12) the couplings $J_b^{(I)}$ with the constrains (13-14), we explore the function φ_I and the non analytic points of φ_I, if any, will signal some phase transition. The critical behavior of φ_I will be characterized by an equation of the type $G_I(\{z_b^{(I)}\}) = 0$, whose solution is provided only by universal, *i.e.*, not depending on the $\{J_b^{(I)}\}$, quantities. A point on the critical surface Σ_I, solution of the equation $G_I = 0$, will be indicated with $\{w_b^{(I)}\}$. The surface Σ_I represents the boundary of some domain $\mathscr{D}_I$ inside which the high temperature expansion providing φ_I converges. As we shall discuss later, the domain $\mathscr{D}_I$ is a convex set so that, if $|z_b^{(I)}| = |\tanh(\beta J_b^{(I)})| < w_b^{(I)}$ for any b, then $\{z_b^{(I)}\} \in \mathscr{D}_I$.

Our main result concerns φ and its singularities, *i.e.* the phase boundaries of the spin glass model. We find convenient to separate these results in the cases of homogeneous and inhomogeneous measures.

4.1. Case of an homogeneous measure (same disorder for any bond)

Let be $d\mu_b \equiv d\mu$ for any bond b of Γ, so that, according to Eqs. (13) and (14), the related Ising model corresponds to a homogeneous Ising model having a single coupling $J_b^{(I)} \equiv J^{(I)}$ and its critical behavior will be characterized by, at most, two points $w_F^{(I)} > 0$ and $w_{AF}^{(I)} < 0$, if any.

First, let us consider a system having a number of first neighbors per vertex proportional to D. Let be $\beta_c^{(SG)}$ and $\beta_c^{(F/AF)}$, respectively, the solutions of the equations, if any

$$\int d\mu \; \tanh^2(\beta_c^{(SG)} J_b) \;\; = w_F^{(I)}, \qquad D > 2, \qquad (19)$$

$$\int d\mu \; \tanh(\beta_c^{(F/AF)} J_b) = w_{F/AF}^{(I)}, \qquad D > 1, \qquad (20)$$

where F or AF, in the l.h.s and r.h.s. of Eq. (20), are in correspondence. We will show that, asymptotically, at high dimensions D, the critical inverse temperature of the spin glass model, β_c, is given by

$$\beta_c = \min\{\beta_c^{(SG)}, \beta_c^{(F/AF)}\}; \qquad (21)$$

and in the paramagnetic phase the following mapping holds

$$\left|\frac{\varphi - \varphi_{eff}}{\varphi}\right| = O\left(\frac{g(\beta^{-1} - \beta_c^{-1})}{D}\right), \quad \forall \beta \le \beta_c, \qquad (22)$$

where, $g(x)$ is some bounded continuous function of order 1 such that $g(x) \to 0$ for $x \to \infty$; and φ_{eff}, is given by

$$\varphi_{eff} = \frac{1}{l}\varphi_I\left(\int d\mu \; \tanh^l(\beta J_b)\right), \qquad D > 2^{l-1}, \qquad (23)$$

where

$$l = \begin{cases} 2, & \text{if} \quad \left|\varphi_I\left(\int d\mu \; \tanh^2(\beta J_b)|\right) \ge 2|\varphi_I\left(\int d\mu \; \tanh(\beta J_b)\right)|, \\ 1, & \text{if} \quad \left|\varphi_I\left(\int d\mu \; \tanh^2(\beta J_b)|\right) < 2|\varphi_I\left(\int d\mu \; \tanh(\beta J_b)\right)|. \end{cases} \qquad (24)$$

If the system Γ is infinite dimensional only in the large sense (Sec. 3), in the above equations D can be settled as infinite and, in the limit $\beta \to \beta_c^-$, Eq. (22) holds exactly.

4.2. Generalization

The generalization of the above mapping to the case of an arbitrary (not homogeneous) measure $d\mu_b$, useful for example for anisotropic models, in which we have to consider even a given number of bond dependencies, follows straightforward. In this case, the related Ising model is defined by a set of, typically few, independent couplings $\{J_b^{(I)}\}$, trough Eqs. (13-14) and its critical behavior will be fully characterized by the points of Σ_I, solution of the equation $G_I = 0$. Equations (19-24) are generalized as follows.

First, let us consider a system having a number of first neighbors per vertex proportional to D. Let be $\beta_c^{(SG)}$ and $\beta_c^{(F/AF)}$, respectively, solutions of the two set of equations (if any)

$$G_I \left(\{ \int d\mu_b \ \tanh^2(\beta_c^{(SG)} J_b) \} \right) = 0, \qquad D > 2, \tag{25}$$

$$G_I \left(\{ \int d\mu_b \ \tanh(\beta_c^{(F/AF)} J_b) \} \right) = 0, \qquad D > 1. \tag{26}$$

Asymptotically, at sufficiently high dimensions D, the critical inverse temperature of the spin glass model β_c is given by

$$\beta_c = \min\{\beta_c^{(SG)}, \beta_c^{(F/AF)}\}; \tag{27}$$

and in the paramagnetic phase the following mapping holds

$$\left| \frac{\varphi - \varphi_{eff}}{\varphi} \right| = O\left(\frac{g(\beta^{-1} - \beta_c^{-1})}{D} \right), \quad \forall \beta \leq \beta_c, \tag{28}$$

$$\varphi_{eff} = \frac{1}{l} \varphi_I \left(\{ \int d\mu_b \ \tanh^l(\beta J_b) \} \right), \qquad D > 2^{l-1}, \tag{29}$$

where

$$l = \begin{cases} 2, & \text{if} \quad |\varphi_I \left(\{ \int d\mu_b \ \tanh^2(\beta J_b) \} \right)| \geq 2 |\varphi_I \left(\{ \int d\mu_b \ \tanh(\beta J_b) \} \right)|, \\ 1, & \text{if} \quad |\varphi_I \left(\{ \int d\mu_b \ \tanh^2(\beta J_b) \} \right)| < 2 |\varphi_I \left(\{ \int d\mu_b \ \tanh(\beta J_b) \} \right)|. \end{cases} \tag{30}$$

If Γ is infinite dimensional only in the large sense, in the above equations D can be settled as infinite and, in the limit $\beta \to \beta_c^-$, Eq. (28) holds exactly.

4.3. Phase diagram: upper critical surface

If $D \to \infty$ in the strict sense, Eqs. (21-24) and their generalizations, Eqs. (27-30), give the exact free energy in the paramagnetic phase (P); the exact critical paramagnetic-spin glass ($P - SG$), $\beta_c^{(SG)}$, and paramagnetic-F/AF ($P - F/AF$), $\beta_c^{(F/AF)}$, surfaces, the stability of which depends on which of the two ones is the minimum. In the case of a homogeneous measure, the suffix F and AF stay for ferromagnetic and antiferromagnetic, respectively. In the general case, such a distinction is possible only in the positive and negative sectors of Σ_I, whereas, for the other sectors, we use the symbol F/AF only to stress that the transition is not $P - SG$. Finally, the constrains $D > 1$ or $D > 2$ along the equations stress that the mapping is not otherwise defined. If instead $D \to \infty$ only in the large sense, Eqs. (21-24) and their generalizations, Eqs. (27-30), are in general exact only in the limit $\beta \to \beta_c^-$ and give

the exact critical paramagnetic-spin glass ($P-SG$), $\beta_c^{(SG)}$, and paramagnetic-F/AF ($P-F/AF$), $\beta_c^{(F/AF)}$, surfaces.

Notice that at zero field, due to the inequality

$$\varphi \leq \varphi_I\left(\{\int d\mu_b \;\tanh(\beta J_b)\}\right), \tag{31}$$

easily derived from the convexity of the logarithm function and, due to the mapping (28), at least for centered measures, our estimations for the free energy become trivially zero if $D\to\infty$ in the strict sense:

$$\lim_{D\to\infty}\varphi = 0, \quad \text{for} \quad \beta\leq\beta_c, \tag{32}$$

and the basic role of Eqs. (22-24), or (28-30), is that to show how, in this limit, φ approaches zero and which are its singularities.

We stress that the presented result is obtained in a replica approach method but does not rely on any ansatz concerning the choice of finding stationary points. In known examples we have so far considered, Eqs. (21-24) give results coinciding with those obtained in the framework of the standard replica approach equipped with the replica-symmetric ansatz, generally accepted as exact above the critical temperature. We recall that, above the critical temperature, the replica symmetric solution has been proved to be the maximum of the functional appearing in the standard replica approach, only in the SK model and in the "p-spin" models [2]-[6]. We stress also that the mapping is exact for any case in which Γ is infinite dimensional and with arbitrary measures including, for example, generalizations of the SK model, Bethe lattice models, and models defined on generalized tree-like structures. Finally, Eqs. (27-30) are useful for more general models, like anisotropic models and models defined on bipartite lattices.

Several general properties can be immediately derived from the critical conditions, Eqs. (19-23), or their generalization, Eqs. (25-29). In particular we find as corollaries:

i) A critical $\beta_c^{(SG)}$ exists and is finite iff the related Ising model has a finite critical $\beta_c^{(I)}$

ii) If an Ising model on Γ with a homogeneous coupling $J>0$ has a phase transition at some $\beta_c^{(I)}$, then the spin glass on Γ with a homogeneous "plus-minus" measure $d\mu(J_b)/dJ_b=\delta(J-J_b)/2+\delta(J+J_b)/2$, will have a SG transition for β_c^{SG} solution of

$$\tanh^2(\beta_c^{(SG)}J) = \tanh(\beta_c^{(I)}J), \tag{33}$$

and, as a consequence, for the plus-minus measure one has always

$$\beta_c^{(SG)} > \beta_c^{(I)} \tag{34}$$

iii) A critical $\beta_c^{(F/AF)}$, exists and is finite iff for the related Ising model exists a point $\{w_b^{(I)}\}\in\Sigma_I$ such that for $w_b^{(I)}\geq 0$ one has $1-2\int_{-\infty}^0 d\mu_b(J_b)\geq w_b^{(I)}$, and for $w_b^{(I)}<0$ one has $1-2\int_{-\infty}^0 d\mu_b(J_b)\leq w_b^{(I)}$, $\forall b\in\Gamma$

iv) In the space of the parameters of the probability distribution $d\mu$ (for simplicity here we consider only the case of a same disorder for any bond), possible lines of coexistence $SG-F$ or $SG-AF$, must intersect the *multicritical* points satisfying the systems of equations given respectively by

$$\left\{\begin{array}{l} \int d\mu \;\tanh^2(\beta_c J_b) = w_F^{(I)}, \\ \int d\mu \;\tanh(\beta_c J_b) = w_F^{(I)}; \end{array}\right. \tag{35}$$

$$\begin{cases} \int d\mu \ \tanh^2(\beta_c J_b) = w_F^{(I)}, \\ \int d\mu \ \tanh(\beta_c J_b) = w_{AF}^{(I)}. \end{cases} \tag{36}$$

4.4. Analytic continuations: coexistence surfaces and correlation functions

The statements until now presented are exact. In this subsection we include the following by-product. For $D=\infty$, below the critical temperature, Eqs. (22-24) or (28-30) are no more valid. This limit has the peculiar feature that, despite the fact that above the critical temperature the free energy is exact, in general, it cannot be analytically continued to lower temperatures [15] and this happens due to the fact that when $D=\infty$, below the critical temperature the related Ising model is ill defined or, in other words, its thermodynamic limit does not exist; the density energy being infinite for any non zero value of the mean magnetization.

What we argue, instead, is that the analytic continuation of some physical quantities below β_c^{-1}, even if it is not rigorous, provides a certain effective approximation. This is in particular the case for the coexistence equations (35-36) for evaluating the spin glass-ferromagnetic, $(SG-F)$, and the spin glass-antiferromagnetic, $(SG-AF)$, boundaries. As we shall show in the examples we will consider, these crossovers are, roughly speaking, close to the Almeida Thouless lines. Due the fact that the free energy provided by the mapping is eaxct in all the paramagnetic phase as $D\to\infty$ in the strict sense, we expect that the above analytic continuation turns out to be better in these cases, rather than in the cases in which $D\to\infty$ only in the large sense.

Similar comments hold for extending the free energy part φ to the case of arbitrary external fields h_i. As we will discuss in Appendix A, the natural extension for φ consists simply in adding the furhter set of arguments $\{\tanh(\beta h_i)\}$ as

$$\varphi_{eff} = \frac{1}{l}\varphi_I\left(\{\int d\mu \ \tanh^l(\beta J_b)\}; \{\tanh(\beta h_i)\}\right), \tag{37}$$

where l is to be determined with the analogous of Eqs. (24).

Now, if in infinite dimensions this extended mapping holds, due to the arbitrariness of the external fields h_i in φ, by derivation we see that we can even calculate a given connected correlation function g starting from the knowledge of the corresponding connected correlation function g_I of the related Ising model. Hence, for example, in infinite dimension, for a correlation function of order two at zero external field, we have

$$g^{(2)}(i_1,i_2) \equiv \int d\mathcal{P}\left(\{J_b\}\right)\left(\langle\sigma_{i_1}\sigma_{i_2}\rangle - \langle\sigma_{i_1}\rangle\langle\sigma_{i_2}\rangle\right) = g_{eff}^{(2)}(i_1,i_2), \tag{38}$$

where $g_{eff}^{(2)}(i_1,i_2)$ is builded as in Eq. (24) trough the correlation function $g_I^{(2)}(i_1,i_2)$

$$g_I^{(2)}(i_1,i_2) \equiv \langle\sigma_{i_1}\sigma_{i_2}\rangle_I - \langle\sigma_{i_1}\rangle_I\langle\sigma_{i_2}\rangle_I . \tag{39}$$

More in general, a connected correlation function of order k of the spin glass model is given by

$$g^{(k)}(i_1,\ldots,i_k) \equiv \int d\mathcal{P}\left(\{J_b\}\right)\langle\sigma_{i_1}\ldots\sigma_{i_k}\rangle^{(c)} = g_{eff}^{(k)}(i_1,\ldots,i_k), \tag{40}$$

where $\langle\sigma_{i_1}\ldots\sigma_{i_k}\rangle^{(c)}$ represents a connected correlation function of order k (see *e.g.* [16]), and $g_{eff}^{(k)}(i_1,\ldots,i_k)$ is given through $g_I^{(k)}(i_1,\ldots,i_k)$ with the analogous of Eqs.

(23-24); $g_I^{(k)}(i_1,\ldots,i_k)$ being the connected correlation function of the corresponding related Ising model

$$g_I^{(k)}(i_1,\ldots,i_k) \equiv \langle \sigma_{i_1}\ldots\sigma_{i_k}\rangle_I^{(c)}. \tag{41}$$

In this work we will not discuss the quadratic correlation functions

$$\int d\mathcal{P}\left(\{J_b\}\right)\left(\langle \sigma_{i_1}\ldots\sigma_{i_k}\rangle^{(c)}\right)^2. \tag{42}$$

5. Disordered systems vs n interacting Ising like systems

As is known, a way to carry out Eq. (5) even with a fixed value of N (for shortness we will always omit the dependence on N), consists in applying the replica method

$$\int d\mathcal{P}\left(\{J_b\}\right)\log\left(Z\left(\{J_b\}\right)\right) = \lim_{n\to 0}\frac{\int d\mathcal{P}\left(\{J_b\}\right)Z^n - 1}{n} = \lim_{n\to 0}\frac{Z^{(n)}-1}{n}, \tag{43}$$

where we have introduced the notation

$$Z^{(n)} \equiv \int d\mathcal{P}\left(\{J_b\}\right)Z\left(\{J_b\}\right)^n. \tag{44}$$

Unlike Z^n, $Z^{(n)}$ represents the partition function of n interacting Ising like systems, *i.e.*, a system with n families of Ising spins $\sigma^{(1)},\ldots,\sigma^{(n)}$ interacting each one through the usual two-body Ising like interaction and, among themselves, through a non quadratic additional interaction. For example for a Gaussian measure

$$\frac{d\mu\left(J_b\right)}{dJ_b} = \frac{1}{\sqrt{2\pi\tilde{J}^2}}\exp\left(-\left(J_b-J_0\right)^2/\tilde{J}^2\right), \tag{45}$$

it is easy to recognize that

$$Z^{(n)} = e^{\frac{\beta^2}{2}\sum_\alpha\sum_b\tilde{J}^2}\times\sum_{\{\sigma_b^{(1)},\ldots,\sigma_b^{(n)}\}} e^{-\beta\sum_\alpha\sum_b j_0\tilde{\sigma}_b^{(\alpha)}+\frac{\beta^2}{2}\sum_{\alpha\neq\beta}\sum_b\tilde{J}^2\tilde{\sigma}_b^{(\alpha)}\tilde{\sigma}_b^{(\beta)}}, \tag{46}$$

where the replica indices α and β run over $1,\ldots,n$. A part from constant factors, the case $n=2$ of Eq. (46) corresponds to the Ashkin-Teller model, for which in two dimensions some exact results are known [17]-[19]. In particular, it is immediate to solve this model when one chooses $J_0=0$, corresponding to the symmetric Gaussian measure. In fact, in this simpler case in Eq. (46) we are left with the only Ising variable $s_b \equiv \tilde{\sigma}_b^{(1)}\tilde{\sigma}_b^{(2)}$ and we are so reduced to a pure Ising sum as follows

$$\sum_{\{\sigma_b^{(1)},\sigma_b^{(2)}\}} f(\cdot) = \sum_{\{\tilde{\sigma}_b^{(1)},\tilde{\sigma}_b^{(2)}\}} f(\cdot), \tag{47}$$

$$\sum_{\{\tilde{\sigma}_b^{(1)},\tilde{\sigma}_b^{(2)}\}} f(\{\tilde{\sigma}_b^{(1)}\tilde{\sigma}_b^{(2)}\}) = 2^{|\Gamma|}\sum_{\{s_b\}} f(\{s_b\}), \tag{48}$$

$|\Gamma|$ being the number of bonds. Note however that for $n>2$ the model $Z^{(n)}$ is no more soluble even for a symmetric measure. This happens due to the impossibility to redefine suitable Ising variables, an aspect related to the concept of frustration. On the other hand, even in the general case, it is always possible to find Ising-like contributions in $Z^{(n)}$. As we shall see later, if the dimension D is sufficiently high, above the critical temperature the Ising-like contributions become the leading part of $Z^{(n)}$.

It is interesting to note here another aspect of the case $n = 2$. It has been well established that the Ashkin-Teller model presents a non universal behavior when the sign in front of the quartic coupling $\tilde{J}^2$ in Eq. (46) is reversed. In that case in fact, the specific heat diverges with a power law whose exponent is a continuous function of the quartic coupling. On the other hand such a non universal behavior cannot appear in the "disordered" system $Z^{(n)}$, simply because the measure (45) is definite only for real values of the disorder parameter $\tilde{J}$. This observation enforce then the idea that at least some features of a spin glass could be effectively described through a suitable Ising model.

6. Representation of $Z(\{J_b\})$ as sum over closed paths

Let us consider a generic Ising model at zero external field with given couplings $\{J_b\}$ defined over some set of links Γ. Note that, since the couplings are arbitrary, what we will say will be valid, in particular, for the related Ising model. It is convenient to introduce the symbol

$$K_b \equiv \beta J_b. \tag{49}$$

For the partition function it holds the so called "high temperature" expansion

$$Z(\{J_b\}) = \prod_{b\in\Gamma} \cosh(K_b) \sum_{\{\sigma_i\}} \prod_{b\in\Gamma} (1 + \tilde{\sigma}_b \tanh(K_b)). \tag{50}$$

As is known the terms obtained by expansion of the product $\prod_{b\in\Gamma}(1 + \tilde{\sigma}_b \tanh(K_b))$, with k bonds proportional to $\tilde{\sigma}_{b_1}\tilde{\sigma}_{b_2}\ldots\tilde{\sigma}_{b_k}$, contribute to the sum over the spins only if the set $\gamma \equiv \{b_1, b_2, \ldots, b_k\}$ constitutes a closed multipoligon over Γ for open or periodic boundary conditions, and a collection of multipoligons and paths, whose end-points belong to the boundary of Γ, for closed conditions (when all the spins on the boundary are fixed to be +1 or -1) (*e.g.* see [20]); in such cases $\tilde{\sigma}_{b_1}\tilde{\sigma}_{b_2}\ldots\tilde{\sigma}_{b_k} \equiv 1$ so that Eq. (50) becomes

$$Z(\{J_b\}) = 2^N \prod_{b\in\Gamma} \cosh(K_b) \sum_{\gamma} \prod_{b\in\gamma} \tanh(K_b), \tag{51}$$

where the sum runs over all the above mentioned paths γ. Note that in the case $\tanh(K_b) = 0$, the sum over the paths gives 1, (*i.e.* the contribution with zero paths must be included). Given the set of couplings $\{J_b\}$, in the thermodynamic limit, the series in the r.h.s. of Eq. (51), normalized to N, will be convergent for values of the parameters $z_b = \tanh(\beta J_b)$ sufficiently small, *i.e.*, inside a suitable set $\mathscr{D}$ whose boundary corresponds to a critical surface Σ of the quenched Ising model. Since in Eq. (51) we have a power series, in z_b, with positive coefficients, it turns out that $\mathscr{D}$, as anticipated, is a convex set.

Let us consider the important case of a homogeneous model: $K_b \equiv K$. In this case, Eq. (51) can be written in the form of a power series in the single parameter $\tanh(K)$

$$Z = 2^N \cosh^{|\Gamma|}(K) \sum_{l=0}^{N} C_l \tanh^l(K), \tag{52}$$

where C_l is the number of paths of length l. Note that, for fixed l, C_l is a function of the size N and $C_l \propto N$, so that in the thermodynamic limit the useful quantity is $c_l \equiv \lim_{N\to\infty} C_l/N$. Whereas C_l represents the total number of paths of length l

in a given lattice of finite size N, c_l represents the total number of paths of length l passing trough a single site in an infinite system; c_l is a growing function of l and for large l this growth is known to be exponential, both for lattice systems and for tree-like structures, see respectively Refs. [21] and [22] and references therein. As c_l is positive, for positive values of K, the series will be convergent and absolutely convergent if $\tanh(K) < w_F^{(I)}$, where

$$\frac{1}{w_F^{(I)}} = \lim_{l\to\infty} c_l^{\frac{1}{l}}, \tag{53}$$

whereas for negative values of K, according to the Leibniz criterion, the series will be convergent if $w_{AF}^{(I)} < \tanh(K)$, where $w_{AF}^{(I)}$ is defined as the largest (in modulo) value of $\tanh(K)$ such that

$$\lim_{l\to\infty} c_l \tanh^l(K) = 0. \tag{54}$$

Hence, for $J > 0$, in the thermodynamic limit the series $\sum_l c_l \tanh^l(\beta J)$ exists for any $\beta < \beta_c^{(F)}$, where $\beta_c^{(F)}$, the inverse of the critical temperature, is solution of the equation $w_F^{(I)} = \tanh(\beta_c^{(F)} J)$, and analogously, for $J < 0$, the series will be convergent for any $\beta < \beta_c^{(AF)}$, where $\beta_c^{(AF)}$, the antiferromagnetic critical temperature, is solution of the equation $w_{AF}^{(I)} = \tanh(\beta_c^{(AF)} J)$. In the thermodynamic limit, given a coupling J, for any value of β lesser than the critical one, it is possible to define the typical length of the path $\bar{l}$ by looking at the distribution $p_l \equiv c_l \tanh(\beta J)^l/(\sum_{l'} c_{l'} \tanh(\beta J)^{l'})$. Note that, as $\beta \to \beta_c^-$, the distribution p_l becomes infinitely flat. Similarly, in the most general case, in which at any bond b we can have a different coupling J_b, the partition function can be rewritten as

$$Z\left(\{J_b\}\right) = 2^N \prod_{b\in\Gamma} \cosh\left(K_b\right) \sum_{\{l_b=0\}}^{N} C(\{l_b\}) \prod_{b\in\gamma} \left(\tanh\left(K_b\right)\right)^{l_b}, \tag{55}$$

where $C(\{l_b\})$ is the number of paths having $\{l_b\}$ units along the bonds $\{b\}$, with $l_b = 0, 1$. Even in this case, from the knowledge of $C(\{l_b\})$, it is possible to calculate the typical length per site $\bar{l}_b$ along the bond b. As in the homogeneous case, also in the general case the critical temperature of the system is determined by the asymptotic behavior of the rescaled coefficients $c(\{l_b\}) \equiv \lim_{N\to\infty} C(\{l_b\})/N$. By re-phrasing in terms of paths, in general we have that:

The critical behavior of the system is determined by the paths of arbitrarily large length

The above observation will be crucial when Γ is a generic graph infinite dimensional only in the large sense.

7. Averaging over the disorder

Let us now introduce the universal function P which takes into account the non trivial part of the high temperature expansion

$$P\left(\{z_b\}\right) \equiv \sum_{\gamma} \prod_{b\in\gamma} z_b, \tag{56}$$

and let us average P over the quenched couplings (the disorder)

$$P^{(1)}\left(\{F_b^{(1)}\}\right) \equiv \int d\mathcal{P}\left(\{J_b\}\right) P\left(\{\tanh(K_b)\}\right), \tag{57}$$

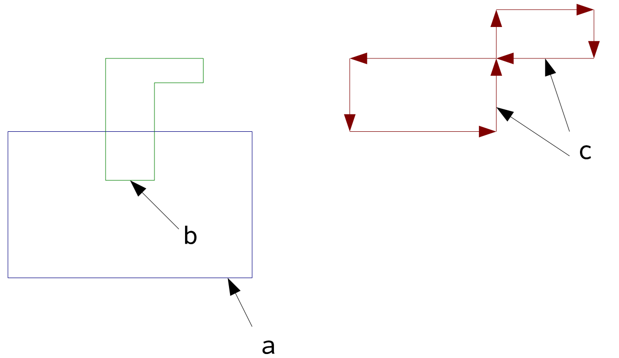

Figure 2. Example of one single path γ on a 2-dimensional lattice. In this case γ is constituted by three sconnected parts: a, b and c. Note that there is no overlapping between bonds. In the part c of the path we have drawn arrows to stress that this part of the path is constituted by only one single connected part, the path c being connected.

where we have introduced

$$F_b^{(1)} \equiv \int d\mu_b \tanh(K_b). \tag{58}$$

From the product nature of the distribution $d\mathcal{P}(\{J_b\})$, Eq. (7), it is immediate to see that $P^{(1)}$ is given in terms of the function P through

$$P^{(1)}\left(\{F_b^{(1)}\}\right) = P\left(\{F_b^{(1)}\}\right) = \sum_{\gamma}\prod_{b\in\gamma} F_b^{(1)}. \tag{59}$$

Later, to evaluate the free energy we will need to consider also the averages of P_I^n for $n \in \mathrm{N}$

$$P^{(n)}\left(\{F_b^{(1)},\ldots,F_b^{(n)}\}\right) \equiv \int d\mathcal{P}(\{J_b\})\, P_I^n\left(\{\tanh(K_b)\}\right), \tag{60}$$

where for $m = 1,\ldots,n$ we have introduced

$$F_b^{(m)} \equiv \int d\mu_b \left(\tanh(K_b)\right)^m. \tag{61}$$

We note that, according to Eqs. (13-14), unlike $P(\{\tanh(K_b)\})$, the function $P\left(\{F_b^{(m)}\}\right)$ is the non trivial part of the high temperature expansion of the related Ising model with couplings $\{F_b^{(m)}\}$.

Let us now generalize Eq. (59) to $P^{(n)}$. From Eqs. (56) and (60) we see that for n integer we can calculate $P^{(n)}$ by summing over n replicas of paths $\gamma_1,\ldots,\gamma_n$, specifying for any of their bonds how many overlaps are there with all the other paths (see Fig. 3). We arrive then at the following expression

$$\begin{aligned} P^{(n)} = \sum_{\gamma_1,\ldots,\gamma_n} \int d\mathcal{P}(\{J_b\}) & \prod_{b\in\cap_{l=1}^n \gamma_l} \tanh^n(K_b)\times \\ & \prod_{(i_1)} \prod_{b\in\cap_{l=1,l\neq i_1}^n \gamma_l\setminus\gamma_{i_1}} \tanh^{(n-1)}(K_b)\times \\ & \prod_{(i_1,i_2)} \prod_{b\in\cap_{l=1,l\neq i_1,i_2}^n \gamma_l\setminus\left(\gamma_{i_1}\cup\gamma_{i_2}\right)} \tanh^{(n-2)}(K_b)\ldots\times \end{aligned}$$

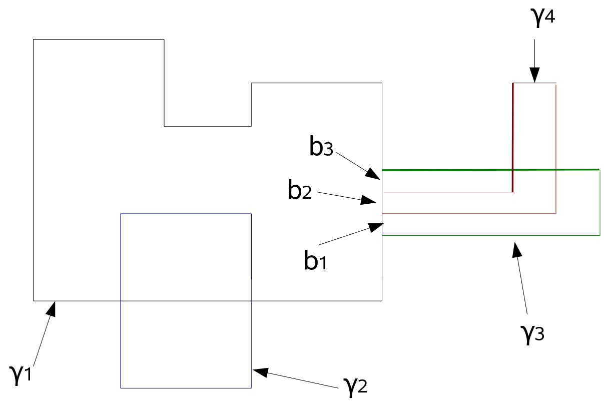

Figure 3. A contribution to the summation of Eq. (62) with $n = 4$. Here we have: a bond with overlap of order 3, $b_2 = \gamma_1 \cap \gamma_3 \cap \gamma_4$; two bonds with overlap of order 2, $b_1 \cup b_3 = \gamma_1 \cap \gamma_3$; and all the other bonds with no overlap (order 1). Note that the paths γ_1 and γ_2 intersect each other as geometrical objects but not as sets (for definition, a path γ is the union of its bonds). The same observation holds for the paths $\gamma_3 \setminus (b_1 \cup b_2 \cup b_3)$ and $\gamma_4 \setminus b_2$.

$$\prod_{(i_n)} \prod_{b\in\gamma_{i_n}\setminus(\cup_{l\neq i_n}\gamma_l)} \tanh\left(K_b\right), \tag{62}$$

where, in the product $\prod_{(i_1,i_2,\ldots i_k)}$, the indeces $i_1, i_2, \ldots, i_k$ run over the $n!/\left((n-k)!k!\right)$ combinations to arrange k numbers from the integers $1, \ldots, n$, and with the symbol (i_1, i_2) we mean the couple i_1, i_2 with $i_1 \neq i_2$ and similarly for $(i_1, i_2, \ldots i_k)$. From Eq. (62) by using Eq. (7) and the definitions (61), we arrive at

$$\begin{aligned} P^{(n)} = & \sum_{\gamma_1,\ldots,\gamma_n} \prod_{b\in\cap_{l=1}^n \gamma_l} F_b^{(n)} \times \\ & \prod_{(i_1)} \prod_{b\in\cap_{l=1,l\neq i_1}^n \gamma_l\setminus\gamma_{i_1}} F_b^{(n-1)} \times \\ & \prod_{(i_1,i_2)} \prod_{b\in\cap_{l=1,l\neq i_1,i_2}^n \gamma_l\setminus\left(\gamma_{i_1}\cup\gamma_{i_2}\right)} F_b^{(n-2)} \times \ldots \times \\ & \prod_{(i_n)} \prod_{b\in\gamma_{i_n}\setminus(\cup_{l\neq i_n}\gamma_l)} F_b^{(1)}. \end{aligned} \tag{63}$$

8. Free energy

From Eq. (51) we have

$$\begin{aligned} \int d\mathcal{P}\left(\{J_b\}\right)\log\left(Z\left(\{J_b\}\right)\right) = & \int d\mathcal{P}\left(\{J_b\}\right)\log\left(2^N \prod_{b\in\Gamma}\cosh(K_b)\right) \\ & + \int d\mathcal{P}\left(\{J_b\}\right)\log\left(\sum_{\gamma}\prod_{b\in\gamma}\tanh(K_b)\right), \end{aligned} \tag{64}$$

from which, by using Eqs. (5) and (7) in the first term of the r.h.s., we get

$$-\beta F = N\log(2) + \sum_{b\in\Gamma}\int d\mu_b \log\left(\cosh(K_b)\right) + \phi, \tag{65}$$

where the non trivial part ϕ is given by

$$\phi \equiv \int d\mathcal{P}\left(\{J_b\}\right)\log\left(\sum_{\gamma}\prod_{b\in\gamma}\tanh(K_b)\right). \tag{66}$$

With the symbol $\phi_I\left(\{z_b^{(I)}\}\right)$ we will mean the non trivial part of the free energy of the related Ising model

$$\phi_I\left(\{z_b^{(I)}\}\right) \equiv \log\left(P_I\left(\{z_b^{(I)}\}\right)\right). \tag{67}$$

The densities of ϕ and ϕ_I will be indicated as φ e φ_I, respectively. The free energy term ϕ will be obtained in terms of $P^{(n)}$, Eq. (60), via the replica method with the analog of Eq. (43):

$$\phi = \lim_{n\to 0}\frac{P^{(n)}-1}{n}. \tag{68}$$

9. Centered measure

In this section we will consider the case of a centered measure:

$$\int d\mu_b(J_b)J_b = 0, \tag{69}$$

which in particular includes the symmetric case:

$$d\mu_b(-J_b) = d\mu_b(J_b). \tag{70}$$

For a centered measure we have

$$F_b^{(2m+1)} = 0. \tag{71}$$

As a consequence, in the high temperature expansion, we find the following structure

$$P^{(0)} = 1, \tag{72}$$

$$P^{(2)} = \sum_{\gamma_1=\gamma_2}\prod_{b\in\gamma_1}F_b^{(2)} = \sum_{\gamma}\prod_{b\in\gamma}F_b^{(2)} = P(\{F_b^{(2)}\}), \tag{73}$$

$$\begin{aligned} P^{(2n)} = & \sum_{\gamma_1,\ldots,\gamma_{2n}\in\mathscr{E}_{2n}} \prod_{b\in\cap_{l=1}^{2n}\gamma_l} F_b^{(2n)} \times \\ & \prod_{(i_1,i_2)} \prod_{b\in\cap_{l=1,l\neq i_1,i_2}^{2n}\gamma_l\setminus\left(\gamma_{i_1}\cup\gamma_{i_2}\right)} F_b^{(2n-2)} \times\ldots\times \\ & \prod_{(i_{n-1},i_n)} \prod_{b\in\gamma_{i_{n-1}}\cap\gamma_{i_n}\setminus\left(\cup_{l\neq i_{n-1},i_n}\gamma_l\right)} F_b^{(2)}, \end{aligned} \tag{74}$$

where $\mathscr{E}_{2n}$ is the set that constrains the sum in the r.h.s. of Eq. (74) to be restricted to combinations of paths for which any bond b may have only an even number of overlaps with any other bond.

Unlike the case of Eq. (73), corresponding to the centered Ashkin-Teller case, we cannot express the general $2n$-th term of Eq. (74) in terms of an Ising like suitable sum as $P(\{F_b^{(m)}\})$. Nevertheless, for any n, in the r.h.s. of Eq. (74) we recognize Ising like contributions such as

$$P^{(4)} = \sum_{\gamma} \prod_{b\in\gamma} F_b^{(4)} + 3 \sum_{\gamma_1,\gamma_2:\gamma_1\cap\gamma_2=\emptyset} \prod_{b\in\gamma_1} F_b^{(2)} \prod_{b\in\gamma_2} F_b^{(2)} + \ldots, \tag{75}$$

$$\begin{aligned} P^{(6)} = & \sum_{\gamma} \prod_{b\in\gamma} F_b^{(6)} + 15 \sum_{\gamma_1,\gamma_2:\gamma_1\cap\gamma_2=\emptyset} \prod_{b\in\gamma_1} F_b^{(2)} \prod_{b\in\gamma_2} F_b^{(4)} \\ & + 15 \sum_{\gamma_1,\gamma_2,\gamma_3:\gamma_i\cap\gamma_j=\emptyset, i\neq j} \prod_{b\in\gamma_1} F_b^{(2)} \prod_{b\in\gamma_2} F_b^{(2)} \prod_{b\in\gamma_3} F_b^{(2)} + \ldots, \end{aligned} \tag{76}$$

and in general we have

$$\begin{aligned} P^{(2n)} = & \sum_{0\leq m_1\leq m_2\ldots\leq m_n:m_1+\ldots m_n=n} C^{(2n)}\left(2m_1,\ldots,2m_n\right) \\ & \times \sum_{\gamma_1,\ldots,\gamma_n:\gamma_i\cap\gamma_j=\emptyset, i\neq j} \prod_{b\in\gamma_1} F_b^{(2m_1)} \cdots \prod_{b\in\gamma_n} F_b^{(2m_n)} + \ldots, \end{aligned} \tag{77}$$

where the coefficient $C^{(2n)}\left(2m_1,\ldots,2m_n\right)$ is given by

$$C^{(2n)}\left(2m_1,\ldots,2m_n\right) = \frac{(2n)!}{\prod_{l=1}^{n}(2m_l)!} \frac{1}{g_1!\cdots g_{n'}!}, \tag{78}$$

where n' is the number of different values of the m's in the sequence $m_1,\ldots,m_n$, and the numbers $g_1,\ldots,g_{n'}$ take into account of the degeneracy of the values $m_1,\ldots,m_n$:

$$g_p = \begin{cases} \sum_{l=1}^{n} \delta_{m_p,m_l}, & m_p \neq 0 \\ 1, & m_p = 0, \end{cases} \tag{79}$$

with $p=1,\ldots,n'$. The number 3 which appears in front of the second term of Eq. (75) counts the number of possible ways to pair (two to two) 4 paths. Similarly in r.h.s. of Eq. (76) the numbers 15 in front of the second and third terms take into account of the possible ways to pair 4 paths and 6 paths, respectively, from a set of 6 paths. Finally, in Eq. (78) we have written the general form of these combinatorial coefficients. However, as we will see, these combinatorial coefficients are of no importance in the thermodynamic limit.

In Eqs. (75-77) we have explicitly written all the contributions in which any path γ coincides with other paths an even number of times and we have then re-named the remaining different paths as similarly done in the third member of Eq. (73): $\gamma \equiv \gamma_1 = \gamma_2$. All these terms, up to the constrain that the re-defined paths γ's cannot have common segments, are Ising like terms, whereas in Eqs. (75-77) we have left dots, $\ldots$, for indicating the other contributions which come from more complicated combinations of paths in which two or more paths γ overlap by themself only partially (*i.e.* not entirely two to two) as shown for example in Fig. 4. These latter contributions cannot be expressed as Ising like terms.

9.1. Approximation in high dimensions

Up to now we have not yet introduced any approximation in our scheme. Unfortunately, the equation (77) for $P^{(2n)}$ presents elements of intractability which avoid an exact calculation even in two dimensions where, on the contrary, the exact

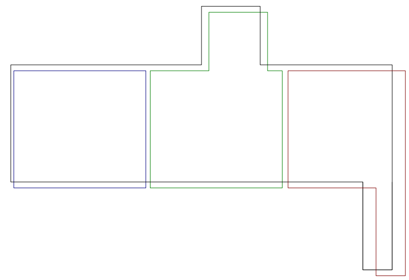

Figure 4. A chain of three connected planar paths encapsulated in a larger planar path. The four paths overlap each other only partially. The paths in the figure are slightly shifted for visual convenience.

solution is possible for the pure Ising model. These difficulties are of two kind. First, in the Ising like terms, the re-named n paths γ cannot have common segments or, in other words, we have to sum over the ensemble of n non overlapping random walks. Second, we do not have any knowledge about the non Ising like terms.

Nevertheless, we can show that, as the dimension D of the system grows, neglecting either the constrain for the non overlapping random walks and the non Ising like terms, implies a smaller and smaller error in evaluating the general expression (77) which, in the limit $D \to \infty$, becomes exact.

In the following we will make use of the fact that for finite and positive values of m we have $O(F_b^{(2m)}) \sim O(F_b^{(2)})^m$. In particular, for the $\pm J$ distribution we have $F_b^{(2m)} = (F_b^{(2)})^m$. We anticipate that, in the limit of infinite dimension, any finite difference between $F_b^{(2m)}$ and $(F_b^{(2)})^m$ becomes irrelevant for the mapping to be valid. For Γ, to be specific, we will consider a D-dimensional hypercube lattice. However, as will be evident later, similar arguments can be repeated for any systems of links Γ infinite dimensional in the large sense (see Sec. 3), as happens, but not only, in the case of generalized tree-like structures.

For the moment, let us consider for simplicity only planar connected paths. Let us start with $n = 4$. We have to sum over all contributions that belong to the set $\mathscr{E}_4$. From the set of all the possible planar connected paths, we have to draw out four arbitrary paths and combine them in any possible way compatible with the set $\mathscr{E}_4$. We find useful to decompose this set as follows

$$\mathscr{E}_4 = \mathscr{E}_{4,4} \cup \mathscr{E}_{2,4} \cup \mathscr{R}_4, \tag{80}$$

where $\mathscr{E}_{4,4}$ is the set of 4 coinciding paths, *i.e.* the set of all one replica paths, see Fig. 5; $\mathscr{E}_{2,4}$ is the set of all paths overlapping in couples (two to two), see Fig. 6; and $\mathscr{R}_4$ is the rest, *i.e.* the set of all paths in $\mathscr{E}_4$ which overlap each other only partially, see Fig. 4.

We now observe that in general, if D is the dimension, and $D > 2$, at any point of our lattice system we have D orthogonal planes where anyone of the 4 planar paths can be arranged. On the other hand, whereas any given element of $\mathscr{E}_{4,4}$ or $\mathscr{R}_4$ lies necessarily on one single plane, we can generate an element of $\mathscr{E}_{2,4}$ by arranging two arbitrary paths, even in two separated planes, for a total of $D(D-1)+D$ combinations. Therefore, we have

$$\frac{|\mathscr{E}_4| - |\mathscr{E}_{2,4}|}{|\mathscr{E}_4|} = O\left(\frac{1}{D-1}\right). \tag{81}$$

Note also that, up to the constrain $\gamma_1 \cap \gamma_2 = \emptyset$, the set $\mathscr{E}_{2,4}$ corresponds to the range of the summation in the second term of Eq. (75).

Analogously, for $n = 6$, we have to consider the following decomposition

$$\mathscr{E}_6 = \mathscr{E}_{6,6} \cup \mathscr{E}_{4,6} \cup \mathscr{E}_{2,6} \cup \mathscr{R}_6, \tag{82}$$

and we see again that the elements of the set $\mathscr{E}_{2,6}$ can be arranged in $D(D-1)(D-2) + O(D(D-1))$ ways, whereas the elements of the other sets can be arranged at most in $D(D-1)$ combinations. Therefore we have

$$\frac{|\mathscr{E}_6| - |\mathscr{E}_{2,6}|}{|\mathscr{E}_6|} = O\left(\frac{1}{D-2}\right). \tag{83}$$

Note that, up to the constrain for the non overlapping of the re-named paths, the set $\mathscr{E}_{2,6}$ corresponds to the range of the summation in the third term of Eq. (76).

We can repeat the above argument for any positive integer n. We always arrive at the conclusion that the Ising like contributions obtained by pairing in all the possible ways two paths are the leading contributions of the set $\mathscr{E}_n$ as follows

$$\frac{|\mathscr{E}_{2n}| - |\mathscr{E}_{2,2n}|}{|\mathscr{E}_n|} = O\left(\frac{1}{D-n+1}\right). \tag{84}$$

Up to the constrain that the re-named paths must not overlap among themselves, the set $\mathscr{E}_{2,2n}$ corresponds to the range of the summation over paths in Eq. (77). On the other hand, by applying the same argument of the dimensionality, we see that neglecting the constrain implies a vanishing error for large D, being

$$\frac{|\mathscr{E}_{2,2n}| - \sum_{\gamma_1,\ldots,\gamma_n:\gamma_i\cap\gamma_j=\emptyset, i\neq j} 1}{|\mathscr{E}_{2,2n}|} = O\left(\frac{1}{D-n+1}\right). \tag{85}$$

Finally, it is not difficult to convince oneself that the a very similar argument can be applied also for general paths, non connected and non planar. In conclusion, taking into account Eqs. (84) and (85), the general expression (77), and the fact that $O(F^{(2m)}) \sim O(F^{(2)})^m$, we arrive at

$$P^{(2n)} = \frac{(2n)!}{2^n n!} P^n\left(\{F_b^{(2)}\}\right) + O\left(\frac{P^{(2n)}}{D-n+1}\right). \tag{86}$$

Furthermore, since for $\beta^{-1} >> (\beta_c^{(I)})^{-1}$ the typical length $\bar{l}(\beta)$ goes to zero, we can improve the description of the above error including an unknown bounded function $g(x)$ of order 1 which goes to zero for $\beta^{-1} >> (\beta_c^{(I)})^{-1}$ and takes into account the fact that in such a limit Eq. (86) becomes a trivial identity, being both $P^{(2n)}$ and $P^n\left(\{F_b^{(2)}\}\right)$ equal to 1 (as we shall see shortly the coefficient in front of P^n does not play any role in the thermodynamic limit). Therefore we can replace Eq. (86) with the finer one

$$P^{(2n)} = \frac{(2n)!}{2^n n!} P^n\left(\{F_b^{(2)}\}\right) + O\left(\frac{P^{(2n)} g(\beta^{-1} - (\beta_c^{(I)})^{-1})}{D-n+1}\right). \tag{87}$$

By exploiting this approximation, we are now able to calculate the free energy term ϕ via the replica formula of Eq. (68) or, more precisely, via the following limit

$$\phi = \lim_{n\to 0} \frac{P^{(2n)} - 1}{2n}. \tag{88}$$

By using the relation

$$(2n)! = 2^n n!(2n-1)!!,$$

and the Gamma function property

$$(2n-1)!! = \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}2^n,$$

form Eq. (87) we arrive at

$$P^{(2n)} = \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}2^n P^n\left(\{F_b^{(2)}\}\right) + O\left(\frac{P^{(2n)}}{D-n}\right). \tag{89}$$

Finally, by taking the derivative with respect to n we get

$$\begin{aligned}\phi = \lim_{n\to 0}\frac{P^{(2n)}-1}{2n} &= \frac{1}{2}\log\left(P\left(F_b^{(2)}\right)\right) + O\left(\frac{\phi}{D}\right)\\ &+ \frac{1}{2}\log(2) + \frac{1}{2}\int_0^\infty dt\frac{\log\left(t\right)}{t^{\frac{1}{2}}e^t},\end{aligned} \tag{90}$$

where we have used

$$\lim_{n\to 0}\frac{\partial}{\partial_n}\log\left(\Gamma\left(n+\frac{1}{2}\right)\right) = \int_0^\infty dt\frac{\log\left(t\right)}{t^{\frac{1}{2}}e^t}. \tag{91}$$

Equation (90) can also be written as

$$\phi = \frac{1}{2}\phi_I\left(\{F_b^{(2)}\}\right) + O\left(\frac{\phi}{D}\right) + \frac{1}{2}\log(2) + \frac{1}{2}\int_0^\infty dt\frac{\log\left(t\right)}{t^{\frac{1}{2}}e^t}. \tag{92}$$

Finally, we note that in the thermodynamic limit the last two terms in the above equation are of no importance, so that, for N large and by using Eq. (65), for the free energy F we find the following expression

$$\begin{aligned}-\beta F = \sum_{b\in\Gamma}&\int d\mu_b \log\left(2\cosh(K_b)\right) + \frac{1}{2}\phi_I\left(\{F_b^{(2)}\}\right)\\ &+ O\left(\frac{\phi}{D}\right), \quad D>2,\end{aligned} \tag{93}$$

$$F_b^{(2)} = \int d\mu_b \tanh^2(K_b), \quad b\in\Gamma. \tag{94}$$

Equation (93-94) has been derived starting from the high temperature expansion of the related Ising model, therefore, taking into account that $\mathscr{D}_I$ is convex, it holds for any temperature β^{-1} such that $\int d\mu_b\ \tanh^2(\beta J_b) \le w_b$. Equation (93-94) tells us that for such temperatures, the free energy of an Ising spin glass model defined over a D-dimensional set of links Γ, with D large and in the presence of a centered disorder, can be effectively expressed in terms of the free energy of an Ising model defined over the same set Γ in which the parameters of the high temperature expansion are replaced by the following effective substitution

$$z_b = \tanh(K_b) \to F_b^{(2)} = \int d\mu_b \tanh^2(K_b), \quad b\in\Gamma. \tag{95}$$

In particular, if the set of equations $\int d\mu_b\ \tanh^2(\beta J_b) = w_b$ admit a solution for some β_c, β_c^{-1} will be the critical temperature of a $P-SG$ transition.

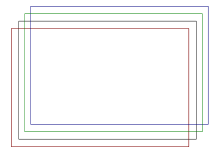

Figure 5. Schematic example of four completely overlapping planar paths. The paths in the figure are slightly shifted for visual convenience.

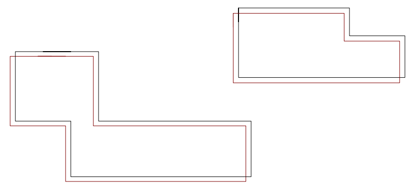

Figure 6. Schematic example of two couples of two completely overlapping planar paths. The paths in the figure are slightly shifted for visual convenience.

10. Generalization to non centered measures

If $F_b^{(2m+1)}$ is no more 0, in constructing $P^{(2n)}$, besides the terms in the set $\mathscr{E}_{2n}$, in which appear only terms involving $F_b^{(2m)}$, $m = 1, \ldots, n$, we have to consider all the other terms belonging to the complement of $\mathscr{E}_{2n}$, $\bar{\mathscr{E}}_{2n}$, *i.e.*, all the families of contributions in which among the $2n$ paths, there is at least one path in which one or more bonds do not overlap any other bond of the other paths or, if there is an overlap, it is an odd overlap, *i.e.*, an odd number of paths overlap over the same bond. Even in this case we can single out the Ising like terms either with even or odd overlaps of the bonds. To this aim we find useful to decompose the whole set of $2n$ paths as:

$$\mathscr{E}_{2n} \cup \bar{\mathscr{E}}_{2n} = \mathscr{F}_0 \cup \ldots \cup \mathscr{F}_k \cup \ldots \cup \mathscr{F}_n \cup \mathscr{R}_n \tag{96}$$

where, for each k, $\mathscr{F}_k$ is the set in which any element is constituted by $2n$ paths in which $2k$ of them overlap each other only an even number of times and the remaining $2n - 2k$ paths overlap each other only an odd number of times, whereas $\mathscr{R}_n$ is the set of all the possible non Ising like terms.

10.1. Approximation in high dimensions

When the dimension D is sufficiently high, in each subset $\mathscr{F}_k$ we can repeat the same argument as in the previous section as follows. Among the $2k$ paths (only even overlaps), the most important terms are those with minimum non zero overlap, *i.e.*, those terms which give a factor $\propto P^k(\{F_b^{(2)}\})$, and, similarly, among the remaining $2n-2k$ paths (only odd overlaps), the most important terms are those with no overlap, *i.e.*, those terms which give a factor $\propto P^{2n-2k}(\{F_b^{(1)}\})$. Note that the proof for the second statement runs as in subsection (9.1), the only difference now is that it works for paths having odd overlaps, so that, in this case, the minimum number of overlapping paths is 1, as opposed to 2 in subsection (9.1), and, as a consequence, the relative error in neglecting the other terms goes as $1/(D - 2n + 1)$, instead of $1/(D - n + 1)$. Furthermore, the minimal dimension to apply the argument for paths with only odd overlaps is not $D > 2$, but $D > 1$.

For what concerns the set $\mathscr{R}_n$, we can neglect these non Ising like contributions also by applying almost the same argument of the subsection (9.1). The only difference now is that these non Ising like terms involve also paths having both even and odd overlaps.

In conclusion, we arrive at the following expression for $P^{(2n)}$

$$P^{(2n)} = \sum_{k=0}^{n} P^k(\{F_b^{(2)}\}) P^{2n-2k}(\{F_b^{(1)}\}) \frac{(2k)!}{2^k k!} \left(\begin{array}{c} 2n \\ 2k \end{array} \right) + O\left(\frac{P^{(2n)}}{D}\right), \tag{97}$$

where the factor $(2k)!/(2^k k!)$, as before, takes into account the number of ways to pair $2k$ paths, whereas the binomial coefficient takes into account the number of ways to choose $2k$ paths from a total of $2n$. Yet, as in the previous section, these combinatorial coefficients are of no importance in the thermodynamic limit, where, both the factors, $P(\{F_b^{(2)}\})$ and $P(\{F_b^{(1)}\})$, grow exponentially in N, so that, in this limit, only the two leading terms, $P^n(\{F_b^{(2)}\})$ and $P^{2n}(\{F_b^{(1)}\})$, are important and, by taking the logarithm of this sum in Eq. (88), the final formulas given in Sec. 4 follow.

11. General graphs

In the previous sections we have proved the mapping by providing a simple argument which applies when the dimension D is sufficiently high. Roughly speaking, the key point is that, if D is the number of axis passing through a site, by choosing at random two of them (or replica), the probability that they coincide (or overlap), goes to zero as $1/D$. For pedagogical reasons we have used this argument by referring to models whose set of links Γ is defined over a D dimensional hypercube lattice Λ, where D is related to the number of first neighbors, $2D$. In these models it is easier to visualize the axis. However, as we will see soon, little changes are involved in the proof if we consider a set Γ infinite dimensional in the large sense. Even for these systems, we can always find the analogous of the number D of independent axis per site and look at the situation when $D \to \infty$.

Let us see now, more specifically, an Ising model defined over a Cayley tree of coordination number $q = k + 1$. It is constructed as follows: one starts from a vertex root '0' and adds q points all connected to 0. This set of q points represents the first shell of q sites. The second shell is instead obtained connecting each one of these sites to new $k = q - 1$ points, and so on for the successive shells (k is then the branching factor).

This lattice is a tree and, therefore, if we do not close in some way the boundaries, or if we do not broke in some minimal way the symmetry up-down, as happens on a hypercube lattice, the non trivial part of the free energy φ_I is zero, and no phase transition is possible. Note that, since a Cayley tree depends heavily and non trivially on the boundary conditions, what is usually studied is a Bethe lattice, consisting in a subset of the Cayley tree infinitely far from the boundary. Yet, even with such a definition, it is known that, unlike the Ising model over the Bethe lattice, the spin glass model over the Bethe lattice remains still dependent on the kind of the boundary coinditions imposed on the sites of the outer part of the Cayley tree. It is known in particular that, even though in this model there is a singularity in the free energy, such a singularity implies a "true spin glass phase transition" (*i.e.*, with the same qualitative picture as the replica symmetry breaking in the SK model [1]) only under certain boundary conditions [23]. However, such a distinction for us is not important;

our aim here is limited to find only the singularities of the model, regardless on the fact that such singularities may or may not imply a true spin glass phase transition.

Let us consider a centered measure and start from the general exact representation (74). As we will see, unlike hypercube lattices, if we analyze the Ising-like contributions for a Bethe lattice, we do not reach exactly the Eqs. (75-77), but a slight modified version of them, which, however, in the thermodynamic limit, coincide. Let us look at the set of all the possible paths starting from the root 0 and arriving at some infinitely far boundary. Note that, in a Bethe lattice, regular or not, such a set coincides with the set of all possible paths. Furhtermore, as will be clear soon, in a Bethe lattice, and more in general in a tree, due to the absence of loops, the effectes of non-Ising like terms are irrelevant, being limited to an overlapping of a finite number of bonds.

It is immediate to see that the term $P^{(2)}$ remains formally as Eq. (73) (see Fig. 7). Let us calculate $P^{(4)}$. Due to the absence of loops, if 4 replica paths starting from 0 coincide for l bonds, after a splitting at the bond $l+1$ in 2 branchs of 2 coinciding paths, the branchs will not overlap each other anymore (see Fig. 8). Therefore, taking into account that the total number of paths of length l starting from 0, is qk^{l-1}, the equivalent of Eq. (75) for a Bethe lattice with n shells becomes

$$P^{(4)} = q\sum_{l=0}^{n} k^l \prod_{r=1}^{l} F_{b_r}^{(4)} 3 \sum_{\gamma_1,\gamma_2} \prod_{b\in\gamma_1\backslash\cup_{r=1}^{l} b_r} F_b^{(2)} \prod_{b\in\gamma_2\backslash\cup_{r=1}^{l} b_r} F_b^{(2)}, \qquad (98)$$

where $\{b_r\}_{r=1}^{n}$, is a sequence of n successive bonds starting from 0. On the other hand, in the thermodynamic limit, $n \to \infty$, the constrains $\backslash \cup_{r=1}^{l} b_r$ in Eq. (98) become negligible and we get

$$P^{(4)} = 3q\sum_{l=0}^{\infty} k^l \prod_{r=1}^{l} F_{b_r}^{(4)} P^2(\{F_b^{(2)}\}). \qquad (99)$$

Similar expressions can be derived for any n. Even if these expressions become quite involved for increasing values of n, in the thermodynamic limit we are always left with a general form of the type

$$P^{(2n)} = B_n P^n(\{F_b^{(2)}\}), \qquad (100)$$

where B_n is a suitable constant which takes into account the combinatorics, Eq. (78), and integers powers of the sums $q\sum_{l=0}^{\infty} k^l \prod_{r=1}^{l} F_{b_r}^{(m)}$, for $4 \leq m \leq n$. Since both of these factors do not grow with the size of the system, as for the previous case of subsection 9.1, we arrive at the same conclusion of Eq. (93) (with $D = \infty$ in this case).

A similar argument can be repeated for tree-like structures and for graphs in which there is at most a finite number of loops per vertex. Unlike the pure Cayley tree now, we have to consider also a finite number of closed paths (see Fig. 9), but the construction of the terms $P^{(2n)}$ runs in the same way. Finally, we see that, as anticipated in Sec. 3, we can consider even more complex non tree-like structures in which the number of loops per vertex is not finite; the only condition we need being that the paths, two at two, share at most a finite number of bonds. The only problem with such complex structures is that the related Ising models are hardly solvable. An exception to this is provided by the fully connected graph, *i.e.*, the SK model.

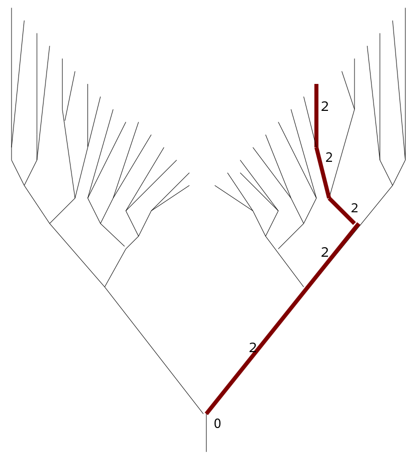

Figure 7. An example of a Bethe lattice with coordination number $q = 3$ ($k = 2$). The path of greater thickness represents two completely overlapping trajectories $\gamma = \gamma_1 = \gamma_2$ going toward the boundary of the lattice.

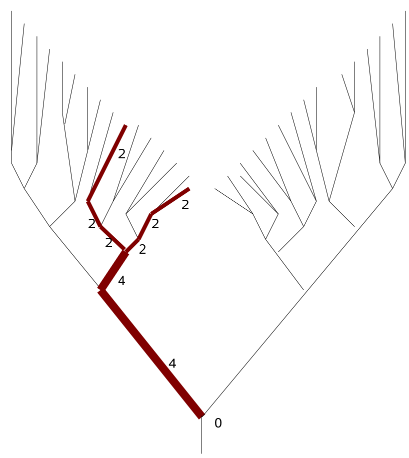

Figure 8. Bethe lattice with $q = 3$ ($k = 2$) with four trajectories overlapping four times along the bonds with label "4" and overlapping 2 to 2 along the bonds with label "2".

12. Applications to finite dimensional models

We now apply the mapping to some finite dimensional cases and compare them with known results. Hereafter we will consider mainly homogeneous models, $d\mu_b \equiv d\mu$, so that, in particular, $F_b^{(2)} \equiv F^{(2)}$. Furthermore in this section we will consider only centered measure and hypercube lattices. For the Ising like term, $\varphi_I(F^{(2)})$, in one and two dimensions, we can use its analytical knowledge coming from the exact solution of the Ising model, whereas, in higher dimensions, we can use for it the knowledge coming from numerical methods. Clearly, since our approach takes into account only the leading term of a $1/D$ expansion, in low dimensional systems, we expect poor results, but it is however interesting to see even in these cases how our approach works. We will force the study even in the cases $D = 1$ and $D = 2$ which can be seen

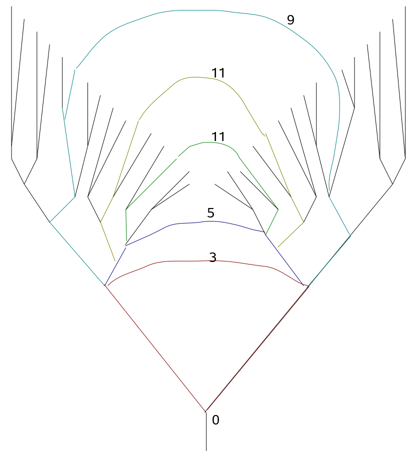

Figure 9. A generalized tree-like structure with loops obtained starting from a a Bethe lattice having previously $q = 3$ ($k = 2$). The drawn loops pass through the root vertex 0. The numbers over the loops are the corresponding lengths.

as singular points where the mapping is not definite.

12.1. One dimensional case

For $D = 1$, if one neglects irrelevant boundary effects, there are no closed paths so that from a direct calculation one sees that φ is exactly 0. The one dimensional case is therefore trivial. Nevertheless, it is worth to observe that, for the same reason, even φ_I is exactly zero, so that for $D = 1$, the mapping, even if not definite, turns out to be exact.

12.2. Two dimensional case

For ϕ_I in two dimensions we can use the Onsager's solution. If we indicate with $z_h = \tanh(k_h)$ and $z_v = \tanh(k_v)$ the horizontal and the vertical parameters of the model, Onsager's formula in the thermodynamic limit gives [25]

$$\varphi_I\left(z_h, z_v\right) = \lim_{L\to\infty} \frac{1}{L}\phi_I\left(z_h, z_v\right) = \lim_{L\to\infty} \frac{1}{L}\log\left(\sum_{\gamma}\prod_{b\in\gamma} z_b\right) =$$
$$\frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi} d\theta_h d\theta_v \log[1 + z_h^2 z_v^2 + z_h^2 + z_v^2 +$$
$$2z_h z_v\left(z_h\cos\left(\theta_v\right) + z_v\cos\left(\theta_h\right)\right) + -2z_h\cos\left(\theta_h\right) - 2z_v\cos\left(\theta_v\right)]. \quad (101)$$

In the case of an isotropic spin glass, we have $d\mu_h \equiv d\mu_v$, so that it is enough to consider the related Ising model with $z \equiv z_h = z_v$, and Eq. (101) simplifies in

$$\varphi_I\left(z\right) = \log\left(\frac{1+z^2}{\sqrt{2}}\right) + \frac{1}{\pi}\int_0^{\frac{\pi}{2}} d\omega \log[1 + \left(1 - \kappa^2\sin^2\left(\omega\right)\right)^{\frac{1}{2}}], \quad (102)$$

where

$$\kappa \equiv 4z\frac{1-z^2}{\left(1+z^2\right)^2}. \quad (103)$$

As is known, $\varphi_I(z)$ is non analytic at $\kappa = \pm 1$, for which the specific heat, as a consequence, has a logarithmic divergence in the temperature. In the high temperature expansion representation, the condition $\kappa = \pm 1$ amounts to $w^{(I)} = \pm(\sqrt{2}-1)$. This implies that in $D = 2$ the pure Ising model presents a ferromagnetic or antiferromagnetic second order phase transition at the critical value $k_c^{(I)} = 0.4407$ ($T_c = 2.269|J|$, if the Boltzmann's constant is taken as 1). From Eq. (93) we see therefore that our approach for a two dimensional Ising spin glass would predict a second order phase transition at β_c solution of

$$\int d\mu \tanh^2(\beta^{(SG)}J) = \pm(\sqrt{2}-1). \tag{104}$$

Of course, this equation may have solutions only if the r.h.s. is positive so that only a correspondence with a "ferromagnetic"-like transition is possible. In particular for the plus-minus distribution (9) centered in J and $-J$, a second order phase transition would take place if $\tanh^2(\beta_c^{(SG)}J) = \sqrt{2}-1$, which has solution for $k_c^{(SG)} = \beta_c^{(SG)}J = 0.7642$ ($T_c^{(SG)} = 1.308J$). This result is in contrast with the (by now) known fact from numerical simulations that, in two dimensions, the Ising spin glass has no phase transition at finite temperature [26]. As is evident from subsection 9.1, $D = 2$ represents a singular case where the mapping is not definite since inside a bidimensional space there is only one plane. This explains why in our approach in two dimensions we find a finite critical temperature; for the two dimensional case, an effective mapping with a suitable Ising model is impossible. The fact that there is not a finite temperature phase transition, implies that in two dimensions the non Ising contributions not only are not negligible, but in the thermodynamic limit constitute the leading part of the high temperature expansion and hide the Ising like effects. Nevertheless, it is interesting to see that some features of this model may be explored even in our Ising-like approach. For example, as appears by comparing the plot of Fig. 10 for the internal energy with the exact numerical results obtained in [27], though the internal energy is wrong, from its slope, away from the critical point, we get a certain evaluation of the specific heat. Note also that, even though it is clearly wrong the result for the critical temperature, according to the general rule (34), in our approximation, the effect of the disorder has however decreased the critical temperature ($T_c^{(SG)} = 1.308J$) with respect to the critical temperature of the pure Ising model ($T_c = 2.269J$).

12.3. Three dimensional case and higher dimensions

For $D > 2$ there is no analytical solution either for the pure Ising model and for the Ising spin glass model. Nevertheless, several very accurate numerical (and partly analytical) data are nowaday available for $D = 3$. From high and low temperature expansions and other numerical techniques it is known that in $D = 3$, the pure Ising model has a critical point at $k_c^{(I)} = 0.221$ ($T_c = 4.51154J$) [28], so that $w^{(I)} = \tanh(k_c^{(I)}) = 0.2180$. Therefore, from the general rule (19), we obtain that in $D = 3$ an Ising spin glass has a phase transition at a critical temperature $\beta_c^{(SG)}$ such that

$$\int d\mu \tanh^2(\beta_c^{(SG)}J) = 0.2180. \tag{105}$$

In particular, for the plus-minus distribution, the above equation gives $k_c^{(SG)} = \beta_c^{(SG)}J = 0.5062$ ($T_c^{(SG)} = 1.975$). On the other hand from [29] a critical temperature

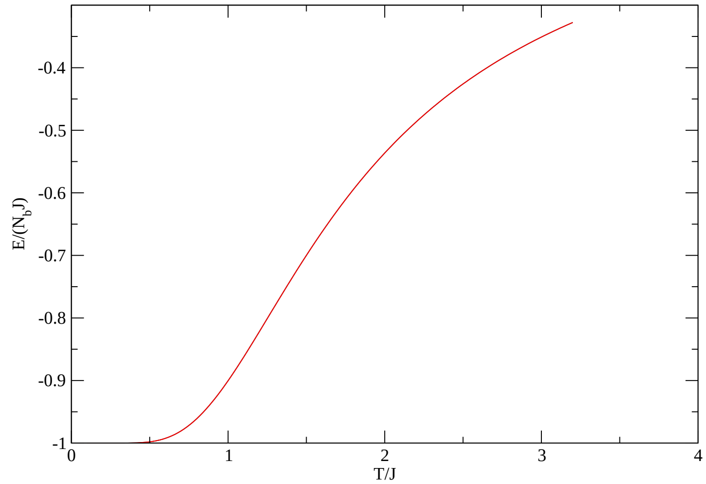

Figure 10. Internal energy normalized per bond and per unit coupling, $e = E/(JN_b)$, for the two dimensional plus-minus spin glass model. The calculation of e is reported in Appendix B.

at $k_c^{(SG)} = 0.851$ ($T_c^{(SG)} = 1.175$) is found.

Similarly, in higher dimensions, we find more and more agreement between our scheme and known results [30, 31, 32], the difference being in good accordance with our estimation of the relative error whose order is expected to be $1/D$.

For sufficiently high, but finite dimensions, we can write a general formula. Let us consider an hypercubic D dimensional spin glass model with a disorder having variance $1/2D$. According to Eqs. (12-14), the related Ising model has the following Hamiltonian

$$H_I = -\sum_{b\in\Gamma} J\tilde{\sigma}_b, \tag{106}$$

so that, if D is large enough it has a critical temperature located a [21]

$$\beta_c^{(I)} J2D = 1 - \frac{1}{2D} + O\left(\frac{1}{D^2}\right). \tag{107}$$

Therefore, by expanding at the first order in $1/D$ Eq. (19), we find that the critical temperature for the corresponding spin glass is given by

$$\beta_c^{(SG)} = 1 + O\left(\frac{1}{D}\right), \tag{108}$$

in accordance with the result of references [10] and [33] in which a perturbative expansion in powers of $1/D$ is performed.

13. Applications to infinite dimensional models

As $D \to \infty$ our scheme becomes exact. In particular, for the SK model, above the critical temperature it reproduces the mean field solution, even though the procedure is completely different from the replica and cavity methods [1]. Notice that, as we have just anticipated in the Sec. 4, our approach, despite of the fact that in $D = \infty$ becomes exact, gives access to the free energy only above the critical temperature and, an

analytic continuation below the critical temperature is not allowed. Note, in particular, that a forced use of the mapping to calculate the free energy at low temperature, would give a completely wrong result. In fact, given a system in D dimensions, it is easy to see that $\varphi_I(\beta)/\beta \to 0$ for $\beta \to \infty$, so that, at zero temperature, from Eq. (65) applied to a plus-minus measure, it remains only $F_0/N = U_0/N = -JD$, whereas the Derrida lower bound for large D gives $U_0/N \sim -J\sqrt{2D\log(2)}$ [34]. However, as just pointed out in the subsection 4.3, we argue that an analytic continuation of other physical quantities, such as the crossover surfaces or the magnetizations, provide a certain effective approximation.

13.1. Sherrington Kirkpatrick model

The SK model corresponds to the spin glass over the fully connected graph and anyone of the N spins interacts with any other spin trough a random coupling J_b whose probability distribution has homogeneous rescaled mean value and variance given respectively by

$$\int d\mu\ J_b = J_0/N, \tag{109}$$

$$\int d\mu\ (J_b - J_0/N)^2 = \tilde{J}^2/N. \tag{110}$$

From Eqs. (13) we see that the Hamiltonian of the related Ising model is

$$H_I = -\sum_{b\in\Gamma_f} J^{(I)}\tilde{\sigma}_b = -\sum_{(i,j)} J^{(I)}\sigma_i\sigma_j, \tag{111}$$

where in the last expression we have rewritten H_I in the usual form as a sum over the couples of sites (i,j). As is well known, for this model, depending on the sign of the coupling $J^{(I)}$, a ferromagnetic-paramagnetic or an antiferromagnetic-paramagnetic phase transition takes place at the same critical temperature given by [35]

$$\beta_c^{(I)}|J^{(I)}|N = 1, \tag{112}$$

which for N large, in terms of the universal quantities $w_{F/AF}^{(I)} = \pm\tanh(\beta_c^{(I)}|J^{(I)}|)$ gives

$$w_{F/AF}^{(I)} = \pm\frac{1}{N} + O\left(\frac{1}{N^3}\right) \tag{113}$$

On the other hand by using Eqs. (109-110) for N large we have

$$\int d\mu\ \tanh^2(\beta J_b) = \frac{\left(\beta\tilde{J}\right)^2}{N} + O\left(\frac{1}{N^3}\right), \tag{114}$$

and

$$\int d\mu\ \tanh(\beta J_b) = \frac{(\beta J_0)}{N} + O\left(\frac{1}{N^3}\right). \tag{115}$$

Therefore, from Eqs. (19) and (20), in the limit $N \to \infty$, we get the following spin glass (SG) and, depending on the sign of J_0, ferromagnetic (F) or antiferromagnetic (AF) phase boundaries

$$\beta_c^{(SG)}\tilde{J} \quad = 1 \tag{116}$$

$$\beta_c^{(F/AF)}J_0 = \pm 1. \tag{117}$$

Finally, according to Eq. (21), by taking the envelope of the curves (116) and (117) we get the upper phase boundaries shown in Fig. 11. In the same figure we report also the coexistence curves $SG-F$ and $SG-AF$ derived from the systems (35-36) whose solution is given by

$$\left(\beta\tilde{J}\right)^2 = \beta|J_0|. \tag{118}$$

Note however that, unlike the upper phase boundaries, the coexistence curves are not exact; they are only representative of the true coexistence curves.

In infinite dimensions, our approach becomes exact only above the upper phase boundaries, where the high temperature part of the free energy, φ, becomes trivially 0. On the other hand, an analytic continuation below these boundaries is not allowed. This fact can be understood considering that the thermodynamic of the related Ising model (111) turns out to be well defined only above the critical temperature, which in the thermodynamic limit is infinite ($\beta_c^{(I)} \to 0$); for a well defined thermodynamic the $J^{(I)}$ in the Hamiltonian (111) should be replaced by $J^{(I)}/N$. Nevertheless, via analytic continuation, besides the extrapolation for the coexistence curves (118), a simple estimation of the Edward Anderson parameters and of the magnetizations is possible. In Appendix C we show that in the SK model, the square root of Edward Anderson parameter and the magnetization, respectively indicated as m_{SG} and m_F, naturally emerge as effective fields in correspondence with the fields of the related Ising model. For $J_0 \geq 0$, they are solution of the following mean field equations, respectively

$$m_{SG} = \tanh\left((\beta\tilde{J})^2 m_{SG}\right), \tag{119}$$

and

$$m_F = \tanh\left(\beta J_0 m_F\right). \tag{120}$$

Note however, that the above expressions are not exact. In particular, the slope of m_{SG} near the critical point is wrong.

Similarly, when $J_0 < 0$ the square root of the Edward Anderson parameter and the magnetization, which now are described by two effective fields, $m^{(a)}$ and $m^{(b)}$ related to the two sublattices a and b in which the initial lattice can be decomposed, are solutions of the following mean field systems, respectively

$$\left\{ \begin{array}{l} m_{SG}^{(a)} = -\tanh\left((\beta\tilde{J})^2 m_{SG}^{(b)}\right), \\ m_{SG}^{(b)} = -\tanh\left((\beta\tilde{J})^2 m_{SG}^{(a)}\right), \end{array} \right. \tag{121}$$

and

$$\left\{ \begin{array}{l} m_{AF}^{(a)} = -\tanh\left(\beta J_0 m_{AF}^{(b)}\right), \\ m_{AF}^{(b)} = -\tanh\left(\beta J_0 m_{AF}^{(a)}\right). \end{array} \right. \tag{122}$$

13.2. SK generalized; random antiferromagnets

One interesting generalization of the SK model, is a model defined over a lattice which can be decomposed in many, say p, sublattices eachone constituted of N sites and with the constrain that a spin over one sublattice interacts only with the spins over

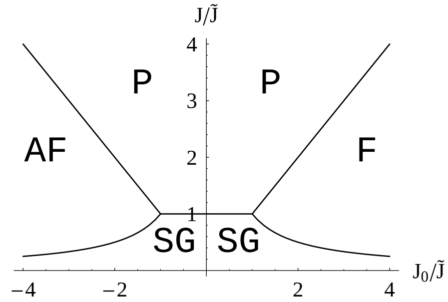

Figure 11. Phase diagram of the Sherrington Kirkpatrick model. The upper line, *i.e.* the envelope of the $P-SG$, the $P-F$ and the $P-AF$ lines, obtained from Eqs. (116) and (117), is exact. The crossover curves $SG-F$ and $SG-AF$ are obtained from Eq. (118).

the other sublattices. Such systems are particularly interesting as models of random antiferromagnets. The Hamiltonian is given by [36]

$$H = -\sum_{(\mu,\nu)}\sum_{(i,j)} J_{(i,j)}^{\mu,\nu}\sigma_{i,\mu}\sigma_{j,\nu}, \tag{123}$$

where the latin indeces $i,j = 1,\dots,N$, label the sites in each sublattice, whereas the greek indeces $\mu,\nu,= 1,\dots,p$, label the sublattices. A coupling $J_{(i,j)}^{\mu,\nu}$ is then in correspondence with the bond $b = (i,\mu;j,\nu)$ connecting the site i of the sublattice μ with the site j of the sublattice ν. Let us consider for simplicity a distribution with a homogeneous rescaled mean value and variance given respectively by

$$\int d\mu\ J_{(i,j)}^{\mu,\nu} = \frac{J_0}{N(p-1)}, \tag{124}$$

$$\int d\mu\ \left(J_{(i,j)}^{\mu,\nu} - \frac{J_0}{N(p-1)}\right)^2 = \frac{\tilde{J}^2}{N(p-1)}. \tag{125}$$

The related Ising model has Hamiltonian

$$H_I = -\sum_{(\mu,\nu)}\sum_{(i,j)} J^{(I)}\sigma_{i,\mu}\sigma_{j,\nu}, \tag{126}$$

Following [37], the most general solution of the related Ising model must be found by introducing p effective fields $m_I^{(1)},\dots,m_I^{(p)}$ satisfying the system of equations

$$m_I^{(l)} = -\sum_{k\neq l}\tanh\left(-\beta m_I^{(k)}J^{(I)}N\right), \quad l = 1,\dots,p. \tag{127}$$

Note that the sign in front of H_I is, for convenience, reversed with respect to [37]; we recall that the coupling $J^{(I)}$ of the related Ising model is homogeneous, but its value can be arbitrary. Note also that, with respect to the convention adopted in [36], the sign for the parameter J_0 has been reversed.

Linearizing Eq. (127) for small fields we arrive at the homogenous system

$$m_I^{(l)} = -x\sum_{k\neq l} m_I^{(k)}, \quad l = 1,\dots,p \tag{128}$$

where x is given by

$$x = -\beta J^{(I)}N. \tag{129}$$

The homogenous system (128) has a non zero solution for values of x for which the matrix of the coefficients A has determinant zero. It is easy to see that this determinant is simple given by

$$\det(A) = (1-x)^{p-1}\left(1+(p-1)x\right), \tag{130}$$

therefore, the system will have a non zero solution for the critical values $x = 1$ or $x = -1/(p-1)$, which using Eq. (129) means

$$-\beta_c^{(I)} J^{(I)} N = \left\{ \begin{array}{l} \frac{-1}{p-1}, \\ 1 \end{array} \right. \tag{131}$$

Recalling the explicit sign in front of the Hamiltonian (126), we note that the first solution corresponds to an ordinary Ising ferromagnet with $J^{(I)} \geq 0$, whereas the second one corresponds to a generalized antiferromagnetic Ising model with $J^{(I)} < 0$ and with p sublattices. For N large, in terms of the universal quantities $\tanh(\beta_c^{(I)} J^{(I)})$, Eq. (131) gives

$$w_F^{(I)} = \frac{1}{N(p-1)} \tag{132}$$

$$w_{AF}^{(I)} = \frac{-1}{N}, \tag{133}$$

which, according to the general rule (19-20) and by using Eqs. (124-125) bring to the following spin glass (SG) and, depending on the sign of J_0, ferromagnetic (F) or antiferromagnetic (AF), phase boundaries

$$\beta_c^{(SG)} \tilde{J} = 1, \tag{134}$$

$$\beta_c^{(F)} J_0 = 1, \quad J_0 \geq 0, \tag{135}$$

$$\beta_c^{(AF)} J_0 = -(p-1), \quad J_0 < 0 \tag{136}$$

Finally, according to Eq. (21), by taking the envelope of the curves (134-136) we get the upper phase boundaries shown in Fig. 12. In the same figure we report also the coexistence curves $SG-F$ and $SG-AF$ derived from Eqs. (35-36) whose equations are given by

$$\left(\beta\tilde{J}\right)^2 = \beta J_0, \quad J_0 \geq 0, \tag{137}$$

for the $SG-F$ crossover, and

$$\left(\beta\tilde{J}\right)^2 = \frac{-\beta J_0}{p-1}, \quad J_0 < 0, \tag{138}$$

for the $SG-AF$ crossover.

Note that for $p = 2$ one recovers the standard SK model.

13.3. Spin glass over Bethe lattice

Let us consider a spin glass model defined over a Bethe lattice of coordination number q (*i.e.*, branching number $k = q-1$, see definition in Sec. 11). In the case of a homogeneous model, $d\mu_b \equiv d\mu$, the related Ising model is the homogeneous Ising model over a regular Bethe lattice with coordination number k for which the exact solution is known [24]. This solution predicts a second order phase transition at a value of $\beta_c^{(I)}$ given by

$$\beta_c^{(I)} J^{(I)} = \frac{1}{2}\log\left(\frac{q}{q-2}\right). \tag{139}$$

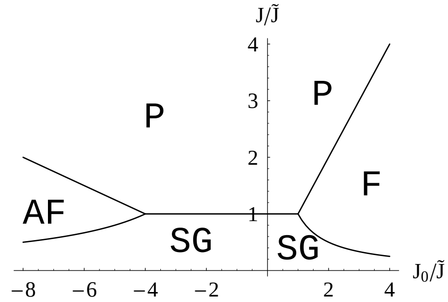

Figure 12. Phase diagram of the generalized Sherrington Kirkpatrick model (123) with $p = 5$. The diagram can be compared with that reported in [36]. The upper line, *i.e.* the envelope of the $P - SG$, the $P - F$ and the $P - AF$ lines, obtained from Eqs. (134-136), is exact. The crossover curves $SG - F$ and $SG - AF$ are obtained from Eqs. (137) and (138).

In terms of the universal quantity $w_F^{(I)} = \tanh(\beta_c^{(I)} J^{(I)})$, Eq. (139) reads

$$w_F^{(I)} = \tanh\left(\beta_c^{(I)} J^{(I)}\right) = \frac{1}{q-1}, \tag{140}$$

where we have made use of the hyperbolic identity

$$\tanh\left(\frac{1}{2}\log(r)\right) = \frac{r-1}{r+1}. \tag{141}$$

Therefore, by using the general rule (19-20) we find that a spin glass (SG) and a disordered ferromagnetic (F) transitions take place at $\beta_c^{(SG)}$ and $\beta_c^{(F)}$, respectively solutions of the two following equations

$$\int d\mu\ \tanh^2\left(\beta_c^{(SG)} J_b\right) = \frac{1}{q-1}, \tag{142}$$

$$\int d\mu\ \tanh\left(\beta_c^{(F)} J_b\right) = \frac{1}{q-1}. \tag{143}$$

Finally, by using Eq. (21) and Eqs. (35-36), the upper phase boundary and the crossover surfaces follow.

14. Conclusions

In the framework of the high temperature expansion, we have derived a general mapping between an Ising spin glass model and a related Ising one, Eqs. (19-24) and (25-30). The mapping is definite above the critical temperature and becomes exat in all the paramagnetic region when the dimension $D \to \infty$ in the strict sense, whereas, more in general, becomes exact in the limit $\beta \to \beta_c^-$ when the dimension $D \to \infty$ in the large sense, where now D, roughly speaking, is the number of independent paths per site (see Sec. 3). The mapping can be applied to find exactly the upper phase boundaries of a spin glass model if the critical temperature of the related Ising model is known. Furthermore, by analytic continutation, the mapping provides even information such as the crossover surfaces, the magnetizations, and the correlation functions. Even if this further application is not exact, we argue that its use can turn out to be quite effective as a first insight to the physics of very complex models.

We have applied the mapping to several known models at finite and infinite dimensions. In particular, the last application of a spin glass model defined over a Bethe lattice suggests a comment. We observe, in fact, that the relations for the Bethe lattice, Eqs. (142-143), have been derived by many authors in several contexts and by using different procedures, see for example [38]-[42]. However, it is quite impressive the easiness by which we have derived these formulas (and similarly those for the SK models); simply starting from the solution of the related Ising model. For example, within the limits that our approach concerns only the singularities of the free energy, we had not to be worried about the rigorous definition of the Bethe lattice. As is known, in fact, the so called Baxter exact solution of the Bethe lattice concerns, more precisely, the inner part of an infinite Cayley tree far, in a sense, from the boundary in which a finite fraction of the total number of spins resides (see Sec. 11). If one does not exclude in some way these boundaries, the model turns out to be greatly sensitive to the boundary conditions and difficult to treat. Another way to avoid this drawback consists in considering for example an ensemble of uncorrelated random graphs in which any site has a mean connectivity equal to q. We stress however that, as far as one considers only the singularities of the free energy, whatever the lattice and the system of links Γ over which we build a spin glass model could be complicated, in our approach, all the mathematical and technical difficulties, even those connected to the delicate questions concerning boundary conditions, if any, are completely reduced at the level of the corresponding related Ising model, which, as a non random model, is remarkable very simpler to solve. It is also clear that this difference becomes particularly important if the given model has many degrees of freedom (mechanical or disorder parameters). In our approach, once the phase boundary of an Ising model in infinite dimension is known, this solution can be immediately applied to find the upper critical boundary of the corresponding spin glass model for which the Ising one turns out to be the related Ising model according to the definition given in Sec. 4. Hence, for example, since the Ising model over a Bethe lattice dos not depend on the chosen boundary conditions, the upper critical surface of an Ising spin glass model over the Bethe lattice will be the same for any chosen boundary condition as well, regardeless of the fact that the non paramagnetic regions can be different (see comment in Sec. 11).

We want to poin out also the numerical advantage that our mapping implies. In fact, even if the analytical knowledge of the critical temperature of some Ising model in not known, an its numerical evaluation turns out to be hugely easier than a direct numerical evaluation for the corresponding Ising spin glass model. Once a numerical estimation for the critical temperature $1/\beta_c^{(I)}$ is known, the mapping returns immediatly the upper critical surface of the spin glass.

The generalization of the mapping to include also a randomness of the set of links, necessary to study random models on random graphs (see, *e.g.*, [43]), will be presented in a forthcoming work (part II). Finally, we observe that the simple argument we have used to prove our mapping, "random Ising system " $\to$ "non random Ising system", which is exact in infinite dimensions, does not seem to be peculiar of the Ising model, so that, even for more general models, a mapping at high temperature "random system" $\to$ "non random system" appears possible.

Acknowledgments

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Appendix A. Free energy in the presence of external fields

The result of Sec. 9 can be formally generalized for arbitrarly external fields $\{h_i\}$. In this case, Eq. (50) is to be modified as follows

$$\begin{aligned} Z\left(\{J_b\};\{h_i\}\right) &= \prod_{b\in\Gamma}\cosh\left(K_b\right)\prod_{i=1}^{N}\cosh\left(H_i\right) \\ &\times\sum_{\{\sigma_i\}}\prod_{b\in\Gamma}\left(1+\tilde{\sigma}_b\tanh\left(K_b\right)\right)\prod_{i=1}^{N}\left(1+\sigma_i\tanh\left(H_i\right)\right), \end{aligned} \tag{A.1}$$

where we have introduced the symbol

$$H_i = \beta h_i. \tag{A.2}$$

Correspondingly, the high temperature expansion of Z will be a series over arbitrary combinations of closed and open paths. Given one of these generalized paths, the weight of a bond b, as before, is provided by $\tanh(K_b)$, whereas the weight of two extremal points i and j of an open path are provided by $\tanh(H_i)$ and $\tanh(H_j)$, respectively. Taking into account that the factors $\tanh(H_i)$'s are not affected by the integration over the J_b's, the generalization of Eq. (63) to generic external field follows straightforward, where, along with the bond-weights $F_b^{(p)}$, now appear also the point-weights $\tanh^q(H_i)$, where p and q count, respectively, the number of overlapping bonds and points, among the n replica generalized paths. On the other hand, it is not difficult to see that we can apply the argument on the dimensionality even to open paths and to combinations of these with closed paths so that even in the presence of an external field the mapping of Eqs. (22-24) follows. The problem here is that this argument is only formal. In fact, if a nonzero magnetic field is present, with respect to the parameters $z_b = \tanh(K_b)$, the high temperature expansion in general can have a zero radius of convergence when $D = \infty$, and this is the case for the related Ising model; for $h_i \neq 0$ one has in general a non zero mean magnetization, so that the density energy of the related Ising model goes to infinite as $D \to \infty$.

Appendix B. Internal energy for the $D = 2$ spin glass

In this appendix we evaluate the free energy and the internal energy for the two dimensional spin glass with a plus-minus measure. As stressed in Sec. (12.2) we get only a rough approximation, but it provides however a case in which an analytical calculation is possible.

According to the general rule of Eqs. (22-24) which, specifically, for a centered measure amount to Eqs. (93-94), the free energy is readily obtained simply by using for φ Eqs. (102) and (103) with the mapping

$$\varphi_I\left(\tanh(\beta J)\right) \to \varphi = \frac{1}{2}\varphi_I\left(\tanh^2(\beta J)\right). \tag{B.3}$$

The internal energy U can be derived from the free energy F by using the formula

$$U = \frac{\partial(\beta F)}{\partial \beta}. \tag{B.4}$$

Therefore, from Eq. (93) for the free energy per bond and per unit of coupling, $e = E/(JN_b)$, we have

$$e = -\tanh(\beta J) - \frac{1}{2J}\frac{\partial \varphi}{\partial \beta}, \tag{B.5}$$

where the factor 2 in the denominator of the second term takes into account that for any two sites we have a bond ($N_b = N/2$). The derivative of φ can be calculated as follows

$$\frac{\partial \varphi}{\partial \beta} = \frac{\partial \varphi}{\partial z}\frac{\partial z}{\partial \beta}, \tag{B.6}$$

$$\frac{\partial \varphi}{\partial z} = \frac{2z}{1+z^2} - \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \kappa \frac{\partial \kappa}{\partial z} d\omega \frac{\sin^2(\omega)}{1 + \left(1 - \kappa^2 \sin^2(\omega)\right)^{\frac{1}{2}}} \frac{1}{\left(1 - \kappa^2 \sin^2(\omega)\right)^{\frac{1}{2}}}, \tag{B.7}$$

$$\frac{\partial z}{\partial \beta} = 2J \frac{\tanh(\beta J)}{\cosh^2(\beta J)}, \tag{B.8}$$

whereas κ is given by Eq. (103) and its derivative is

$$\frac{\partial \kappa}{\partial z} = 4\frac{1-z^2}{(1+z^2)^2} - 8\frac{z^2}{(1+z^2)^2} - 164 z^2 \frac{1-z^2}{(1+z^2)^3}. \tag{B.9}$$

By inserting the expressions (B.6-B.8) in Eq. (B.5) and by using for φ the mapping (B.3), one gets the free internal energy e whose plot is reported in Fig. (10).

Appendix C. Effective fields

In our approach the key ingredient is the knowledge of the non trivial part of the free energy ϕ_I of the related Ising model, in terms of which the free energy is written as

$$-\beta F_I = N\log(2) + \frac{N(N-1)}{2}\log\left(\cosh\left(\beta J^{(I)}\right)\right) + \phi_I\left(\tanh\left(\beta J^{(I)}\right)\right). \tag{C.10}$$

Once ϕ_I is known, according to Eqs. (23-24), in order to calculate ϕ, we have to implement in ϕ_I the following transformations

$$\tanh\left(\beta J^{(I)}\right) \to \int d\mu\ \tanh^2(\beta J_b) = \frac{\left(\beta \tilde{J}\right)^2}{N} + O\left(\frac{1}{N^3}\right), \tag{C.11}$$

and

$$\tanh\left(\beta J^{(I)}\right) \to \int d\mu\ \tanh(\beta J_b) = \frac{(\beta J_0)}{N} + O\left(\frac{1}{N^3}\right). \tag{C.12}$$

As is well known, up to terms negligible in the thermodynamic limit, the fully connected Ising model with homogeneous coupling $J^{(I)}$, has a mean field solution which, depending on the sign of $J^{(I)}$ can be ferromagnetic or antiferromagnetic. It is convenient to distinguish the two sub-cases.

Let us suppose that $J^{(I)} \geq 0$. In this case the mean field solution for the related Ising model is given by

$$-\beta F_I = N\log\left(2\cosh\left(\beta J^{(I)} m_I N\right)\right) - \frac{N^2 \beta J^{(I)} m_I^2}{2}, \tag{C.13}$$

where m_I satisfies the mean field equation

$$m_I = \tanh\left(\beta m_I J^{(I)} N\right). \tag{C.14}$$

From Eqs. (C.10) and (C.13-C.14), for the density $\varphi_I = \phi_I/N$ we then get

$$\begin{aligned}\varphi_I\left(\tanh\left(\beta J^{(I)}\right)\right) &= \log\left(\cosh\left(\beta J^{(I)} m_I N\right)\right) - \frac{N}{2}\log\left(\cosh\left(\beta J^{(I)}\right)\right) \\ &\quad - \left.\frac{\beta J^{(I)} m_I^2 N}{2}\right|_{m_I = \tanh\left(\beta m_I J^{(I)} N\right)}\end{aligned} \tag{C.15}$$

Equation (C.15) must now be used in the mapping of Eqs. (23-24), by using the transformations (C.11-C.12). To this aim it is convenient to use the relation $\cosh^2(x) = 1/(1-\tanh^2(x))$ and to rewrite Eq. (C.15) as

$$\begin{aligned}\varphi_I\left(\tanh\left(\beta J^{(I)}\right)\right) &= -\frac{1}{2}\log\left(1-m_I^2\right) + \frac{N}{4}\log\left(1-\tanh^2\left(\beta J^{(I)}\right)\right) \\ &\quad - \left.\frac{\operatorname{arctanh}\left(\tanh\left(\beta J^{(I)}\right)\right) m_I^2 N}{2}\right|_{m_I = \tanh\left(\operatorname{arctanh}\left(\tanh\left(\beta J^{(I)}\right)\right) m_I N\right)}\end{aligned} \tag{C.16}$$

By inserting the transformations (C.11) and (C.12) in Eq. (C.16) for N large we arrive, respectively, at

$$\varphi_I\left(\int d\mu\ \tanh^2(\beta J)\right) = -\frac{1}{2}\log\left(1-m_{SG}^2\right) - \frac{\left(\beta\tilde{J}\right)^2 m_{SG}^2}{2}, \tag{C.17}$$

and

$$\varphi_I\left(\int d\mu\ \tanh(\beta J)\right) = -\frac{1}{2}\log\left(1-m_F^2\right) - \frac{\beta J_0 m_F^2}{2}, \tag{C.18}$$

where the effective fields m_{SG} and m_F have been obtained, respectively, by using the transformations (C.11) and (C.12) and are given by

$$m_{SG} = \tanh\left((\beta\tilde{J})^2 m_{SG}\right), \tag{C.19}$$

and

$$m_F = \tanh\left(\beta J_0 m_F\right). \tag{C.20}$$

The obvious interpretation of these fields is that, whereas m_F is the magnetization of the system (in presence of a disorder with variance $\tilde{J}$), m_{SG} represents $\sqrt{q_{EA}}$, *i.e.*, up to a square root, it is the Edward-Anderson order parameter, which below the critical temperature is non zero even for $J_0 = 0$.

Let us now suppose $J^{(I)} < 0$. In this case the mean field solution for the related Ising model is given in terms of two effective fields $m_I^{(a)}$ and $m_I^{(b)}$ related to two sublattices a and b in which the given lattice Λ can be decomposed. The sublattices a and b are defined symmetrically so that the first neighbors of a site of the sublattice a are sites of the sublattice b and vice versa. In terms of the effective fields $m_I^{(a)}$ and $m_I^{(b)}$ the free energy of the related Ising model is given by

$$\begin{aligned}-\beta F_I &= \frac{N}{2}\log\left(2\cosh\left(\beta J^{(I)} m_I^{(a)} N\right) 2\cosh\left(\beta J^{(I)} m_I^{(b)} N\right)\right) \\ &\quad + \frac{N^2 \beta J^{(I)} m_I^{(a)} m_I^{(b)}}{2},\end{aligned} \tag{C.21}$$

where $m_I^{(a)}$ and $m_I^{(b)}$ satisfy the mean field system

$$\left\{ \begin{array}{l} m_I^{(a)} = -\tanh\left(\beta m_I^{(b)} J^{(I)} N\right), \\ m_I^{(b)} = -\tanh\left(\beta m_I^{(a)} J^{(I)} N\right). \end{array} \right. \tag{C.22}$$

Similarly to what done in the previous case, from Eqs. (C.10) and (C.21-C.22), for the density φ_I we get

$$\begin{aligned} \varphi_I\left(\tanh\left(\beta J^{(I)}\right)\right) &= \frac{1}{2}\log\left(\cosh\left(\beta J^{(I)} m_I^{(a)} N\right)\cosh\left(\beta J^{(I)} m_I^{(b)} N\right)\right) \\ &+ \frac{N\beta J^{(I)} m_I^{(a)} m_I^{(b)}}{2} - \frac{N}{2}\log\left(\cosh\left(\beta J^{(I)}\right)\right), \end{aligned} \tag{C.23}$$

and by inserting the transformations (C.11) and (C.12) for N large we arrive, respectively, at

$$\begin{aligned} \varphi_I\left(\int d\mu\ \tanh^2(\beta J)\right) &= -\frac{1}{4}\log\left(\left(1 - {m_{SG}^{(a)}}^2\right)\left(1 - {m_{SG}^{(b)}}^2\right)\right) \\ &+ \frac{\left(\beta\tilde{J}\right)^2 m_{SG}^{(a)} m_{SG}^{(b)}}{2}, \end{aligned} \tag{C.24}$$

and

$$\begin{aligned} \varphi_I\left(\int d\mu\ \tanh(\beta J)\right) &= -\frac{1}{4}\log\left(\left(1 - {m_{AF}^{(a)}}^2\right)\left(1 - {m_{AF}^{(b)}}^2\right)\right) \\ &+ \frac{\beta J_0 m_{AF}^{(a)} m_{AF}^{(b)}}{2}, \end{aligned} \tag{C.25}$$

where the effective fields $m_{SG}^{(a)}$ and $m_{SG}^{(b)}$, and $m_F^{(a)}$ and $m_F^{(b)}$, satisfy the following mean field systems, respectively

$$\left\{ \begin{array}{l} m_{SG}^{(a)} = -\tanh\left((\beta\tilde{J})^2 m_{SG}^{(b)}\right), \\ m_{SG}^{(b)} = -\tanh\left((\beta\tilde{J})^2 m_{SG}^{(a)}\right), \end{array} \right. \tag{C.26}$$

and

$$\left\{ \begin{array}{l} m_{AF}^{(a)} = -\tanh\left(\beta J_0 m_{AF}^{(b)}\right), \\ m_{AF}^{(b)} = -\tanh\left(\beta J_0 m_{AF}^{(a)}\right). \end{array} \right. \tag{C.27}$$

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